Problem 18

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

\[ x^2(1 - x)^2y'' + 2xy' + 4y = 0 \]

Solution

The coefficient of \( y'' \) has zeros at \( x = 0 \) and \( x = 1 \), which means \( x = 0 \) and \( x = 1 \) are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by \( x^2(1 - x)^2 \)

\[ y'' + \frac{2}{x(1 - x)^2}y' + \frac{4}{x^2(1 - x)^2}y = 0 \]

and compute the following limits.

\[
\begin{align*}
\lim_{x \to 0} x^{-2} \frac{2}{x(1 - x)^2} &= \lim_{x \to 0} \frac{2}{(1 - x)^2} = 2 \\
\lim_{x \to 0} x^{-2} \frac{4}{x^2(1 - x)^2} &= \lim_{x \to 0} \frac{4}{(1 - x)^2} = 4 \\
\lim_{x \to 1} (x - 1)^{-2} \frac{2}{x(1 - x)^2} &= \lim_{x \to 1} \frac{2}{x(1 - x)} = \infty \\
\lim_{x \to 1} (x - 1)^{-2} \frac{4}{x^2(1 - x)^2} &= \lim_{x \to 1} \frac{4}{x^2} = 4
\end{align*}
\]

Because both limits as \( x \to 0 \) are finite, \( x = 0 \) is a regular singular point. However, because one of the limits as \( x \to 1 \) is infinite, \( x = 1 \) is an irregular singular point.