Problem 2

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

\[(x + 1)^2 y'' + 3(x + 1)y' + 0.75y = 0\]

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form \(y = (x + 1)^r\).

\[y = (x + 1)^r \rightarrow y' = r(x + 1)^{r-1} \rightarrow y'' = r(r - 1)(x + 1)^{r-2}\]

Substitute these expressions into the ODE.

\[(x + 1)^2 r(r - 1)(x + 1)^{r-2} + 3(x + 1)r(x + 1)^{r-1} + 0.75(x + 1)^r = 0\]

\[r(r - 1)(x + 1)^r + 3r(x + 1)^r + 0.75(x + 1)^r = 0\]

Divide both sides by \((x + 1)^r\).

\[r(r - 1) + 3r + 0.75 = 0\]

Solve for \(r\).

\[r^2 + 2r + 0.75 = 0\]

\[(r + 1.5)(r + 0.5) = 0\]

\[r = \{-1.5, -0.5\}\]

Two solutions to the ODE are \(y = (x + 1)^{-1.5}\) and \(y = (x + 1)^{-0.5}\). According to the principle of superposition, the general solution is a linear combination of these two. Therefore,

\[y(x) = C_1(x + 1)^{-1.5} + C_2(x + 1)^{-0.5}\]