Problem 23

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

\[(x + 3)y'' - 2xy' + (1 - x^2)y = 0\]

Solution

The coefficient of \(y''\) has a zero at \(x = -3\), which means \(x = -3\) is a singular point. To determine whether it is regular or irregular, divide both sides of the ODE by \(x + 3\)

\[y'' - \frac{2x}{x + 3}y' + \frac{1 - x^2}{x + 3}y = 0\]

and compute the following limits.

\[
\lim_{x \to -3} (x + 3) \left( -\frac{2x}{x + 3} \right) = \lim_{x \to -3} (-2x) = 6 \\
\lim_{x \to -3} (x + 3)^2 \left( \frac{1 - x^2}{x + 3} \right) = \lim_{x \to -3} (x + 3)(1 - x^2) = 0
\]

Because both limits as \(x \to -3\) are finite, \(x = -3\) is a regular singular point.