Problem 25

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

\[(x + 2)^2(x - 1)y'' + 3(x - 1)y' - 2(x + 2)y = 0\]

Solution

The coefficient of \(y''\) has zeros at \(x = 1\) and \(x = -2\), which means \(x = 1\) and \(x = -2\) are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by \((x + 2)^2(x - 1)\)

\[y'' + \frac{3}{(x + 2)^2}y' - \frac{2}{(x + 2)(x - 1)}y = 0\]

and compute the following limits.

\[
\lim_{x \to 1} (x - 1) \left( \frac{3}{(x + 2)^2} \right) = \lim_{x \to 1} \frac{3(x - 1)}{(x + 2)^2} = 0
\]

\[
\lim_{x \to 1} (x - 1)^2 \left[ -\frac{2}{(x + 2)(x - 1)} \right] = \lim_{x \to 1} \frac{-2(x - 1)}{(x + 2)(x - 1)} = 0
\]

\[
\lim_{x \to -2} (x + 2)^2 = \lim_{x \to -2} \frac{3}{x + 2} = \infty
\]

\[
\lim_{x \to -2} (x + 2)^2 \left[ -\frac{2}{(x + 2)(x - 1)} \right] = \lim_{x \to -2} \frac{-2(x + 2)}{x - 1} = 0
\]

Because at least one of the limits as \(x \to -2\) is infinite, \(x = -2\) is an irregular singular point. However, because both limits as \(x \to 1\) are finite, \(x = 1\) is a regular singular point.