Problem 26

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

\[ x(3 - x)y'' + (x + 1)y' - 2y = 0 \]

Solution

The coefficient of \( y'' \) has zeros at \( x = 0 \) and \( x = 3 \), which means \( x = 0 \) and \( x = 3 \) are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by \( x(3 - x) \)

\[ y'' + \frac{x + 1}{x(3 - x)}y' - \frac{2}{x(3 - x)}y = 0 \]

and compute the following limits.

\[ \lim_{x \to 0} \frac{x + 1}{x(3 - x)} = \lim_{x \to 0} \frac{x + 1}{3 - x} = \frac{1}{3} \]

\[ \lim_{x \to 0} x^2 \left[ -\frac{2}{x(3 - x)} \right] = \lim_{x \to 0} -\frac{2x}{3 - x} = 0 \]

\[ \lim_{x \to 3} (x - 3) \frac{x + 1}{x(3 - x)} = -\lim_{x \to 3} \frac{x + 1}{x} = -\frac{4}{3} \]

\[ \lim_{x \to 3} (x - 3)^2 \left[ -\frac{2}{x(3 - x)} \right] = \lim_{x \to 3} (3 - x)^2 \left[ -\frac{2}{x(3 - x)} \right] = \lim_{x \to 3} \frac{-2(3 - x)}{x} = 0 \]

Because both limits as \( x \to 0 \) and \( x \to 3 \) are finite, \( x = 0 \) and \( x = 3 \) are regular singular points.