Problem 3

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

\[ x^2y'' - 3xy' + 4y = 0 \]

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form \( y = x^r \).

\[ y = x^r \rightarrow y' = rx^{r-1} \rightarrow y'' = r(r - 1)x^{r-2} \]

Substitute these expressions into the ODE.

\[ x^2(r(r - 1)x^{r-2} - 3rx^{r-1} + 4x^r) = 0 \]

\[ r(r - 1)x^r - 3rx^r + 4x^r = 0 \]

Divide both sides by \( x^r \).

\[ r(r - 1) - 3r + 4 = 0 \]

Solve for \( r \).

\[ r^2 - 4r + 4 = 0 \]

\[ (r - 2)^2 = 0 \]

\[ r = \{2\} \]

One solution to the ODE is \( y = x^2 \). The general solution can be obtained by using the method of reduction of order: Plug in \( y(x) = c(x)x^2 \) into the ODE.

\[ x^2[c(x)x^2]'' - 3x[c(x)x^2]' + 4[c(x)x^2] = 0 \]

Evaluate the derivatives.

\[ x^2[c'(x)x^2 + 2c(x)x]' - 3x[c'(x)x^2 + 2c(x)x] + 4c(x)x^2 = 0 \]

\[ x^2[c''(x)x^4 + 4c'(x)x^3 + 2c(x)x] - 3x[c'(x)x^2 + 2c(x)x] + 4c(x)x^2 = 0 \]

\[ c''(x)x^4 + 4c'(x)x^3 + 2c(x)x - 3c'(x)x^3 - 6c(x)x^2 + 4c(x)x^2 = 0 \]

\[ c''(x)x^4 + c'(x)x^3 = 0 \]

Solve this ODE for \( c(x) \).

\[ \frac{c''(x)}{c'(x)} = -\frac{1}{x} \]

\[ \frac{d}{dx} \ln c'(x) = -\frac{1}{x} \]

Integrate both sides with respect to \( x \).

\[ \ln c'(x) = -\ln x + C_1 \]
Exponentiate both sides.
\[ c'(x) = e^{-\ln x + C_1} \]
\[ = e^{\ln x - 1 + C_1} \]
\[ = e^{\ln x - 1} e^{C_1} \]
\[ = x^{-1} e^{C_1} \]
\[ = \frac{e^{C_1}}{x} \]

Integrate both sides with respect to \( x \) once more.
\[ c(x) = e^{C_1} \ln x + C_2 \]

Therefore, using a new constant \( C_3 \) for \( e^{C_1} \),
\[ y(x) = c(x)x^2 \]
\[ = (C_3 \ln x + C_2)x^2 \]
\[ = C_3 x^2 \ln x + C_2 x^2. \]