Problem 9

In each of Problems 1 through 12, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

\[ x^2y'' - 5xy' + 9y = 0 \]

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form \( y = x^r \).

\[ y = x^r \rightarrow y' = rx^{r-1} \rightarrow y'' = r(r-1)x^{r-2} \]

Substitute these expressions into the ODE.

\[
x^2r(r-1)x^{r-2} - 5rx^{r-1} + 9x^r = 0
\]

\[
r(r-1)x^r - 5rx^r + 9x^r = 0
\]

Divide both sides by \( x^r \).

\[
r(r-1) - 5r + 9 = 0
\]

Solve for \( r \).

\[
r^2 - 6r + 9 = 0
\]

\[
(r - 3)^2 = 0
\]

One solution to the ODE is \( y = x^3 \). The general solution can be obtained by using the method of reduction of order: Plug in \( y(x) = c(x)x^3 \) into the ODE.

\[
x^2[c(x)x^3]'' - 5x[c(x)x^3]' + 9[c(x)x^3] = 0
\]

Evaluate the derivatives.

\[
x^2[c'(x)x^3 + 3c(x)x^2]' - 5x[c'(x)x^3 + 3c(x)x^2] + 9[c(x)x^3] = 0
\]

\[
x^2[c''(x)x^3 + 6c'(x)x^2 + 6c(x)x] - 5x[c'(x)x^3 + 3c(x)x^2] + 9[c(x)x^3] = 0
\]

\[
c''(x)x^5 + 6c'(x)x^4 + 6c(x)x^3 - 5c'(x)x^4 - 15c(x)x^3 + 9c(x)x^3 = 0
\]

\[
c''(x)x^5 + c'(x)x^4 = 0
\]

Solve this ODE for \( c(x) \).

\[
\frac{c''(x)}{c'(x)} = -\frac{1}{x}
\]

\[
\frac{d}{dx} \ln c'(x) = -\frac{1}{x}
\]

Integrate both sides with respect to \( x \).

\[
\ln c'(x) = -\ln x + C_1
\]
Exponentiate both sides.

\[ c'(x) = e^{-\ln x + C_1} \]
\[ = e^{\ln x^{-1} + C_1} \]
\[ = e^{\ln x^{-1} e^{C_1}} \]
\[ = x^{-1} e^{C_1} \]
\[ = \frac{e^{C_1}}{x} \]

Integrate both sides with respect to \( x \) once more.

\[ c(x) = e^{C_1} \ln x + C_2 \]

Therefore, using a new constant \( C_3 \) for \( e^{C_1} \),

\[ y(x) = c(x)x^3 \]
\[ = (C_3 \ln x + C_2)x^3 \]
\[ = C_3 x^3 \ln x + C_2 x^3. \]