# Problem 12

Recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and that $\sin bt = (e^{ibt} - e^{-ibt})/2i$. In each of Problems 11 through 14, find the Laplace transform of the given function; $a$ and $b$ are real constants. Assume that the necessary elementary integration formulas extend to this case.

$$f(t) = \cos bt$$

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### Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Find the Laplace transform of $\cos bt$.

$$\begin{aligned}
\mathcal{L}\{\cos bt\} &= \int_0^\infty e^{-st}\cos bt\,dt \\
&= \int_0^\infty e^{-st}\frac{e^{ibt}+e^{-ibt}}{2}\,dt \\
&= \frac{1}{2}\int_0^\infty e^{-st}(e^{ibt}+e^{-ibt})\,dt \\
&= \frac{1}{2}\left(\int_0^\infty e^{ibt-st}\,dt + \int_0^\infty e^{-ibt-st}\,dt\right) \\
&= \frac{1}{2}\left[\int_0^\infty e^{(ib-s)t}\,dt + \int_0^\infty e^{(-ib-s)t}\,dt\right] \\
&= \frac{1}{2}\left[\frac{1}{ib-s}e^{(ib-s)t}\Big|_0^\infty + \frac{1}{-ib-s}e^{(-ib-s)t}\Big|_0^\infty\right] \\
&= \frac{1}{2}\left[\frac{1}{ib-s}(-1) + \frac{1}{-ib-s}(-1)\right] \qquad (1) \\
&= \frac{1}{2}\left(\frac{1}{s-ib} + \frac{1}{s+ib}\right) \\
&= \frac{1}{2}\left[\frac{s+ib+s-ib}{(s-ib)(s+ib)}\right] \\
&= \frac{1}{2}\left(\frac{2s}{s^2-i^2b^2}\right) \\
&= \frac{s}{s^2+b^2}
\end{aligned}$$

Note that for equation (1) to hold, it is critical that $s > 0$.