Problem 15

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

$$f(t) = te^{at}$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of te^{at}.

$$\begin{aligned}
\mathcal{L}\{te^{at}\} &= \int_0^\infty e^{-st} te^{at}\, dt \\
&= \int_0^\infty te^{at-st}\, dt \\
&= \int_0^\infty te^{(a-s)t}\, dt \\
&= \int_0^\infty t \frac{d}{dt}\left[\frac{1}{a-s}e^{(a-s)t}\right] dt \\
&= t\left[\frac{1}{a-s}e^{(a-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (1)\left[\frac{1}{a-s}e^{(a-s)t}\right] dt \\
&= -\frac{1}{a-s}\int_0^\infty e^{(a-s)t}\, dt \qquad (1) \\
&= -\frac{1}{(a-s)^2}e^{(a-s)t}\Bigg|_0^\infty \\
&= -\frac{1}{(a-s)^2}(-1) \qquad (2) \\
&= \frac{1}{(a-s)^2} \\
&= \frac{1}{(s-a)^2}
\end{aligned}$$

Note that for equations (1) and (2) to hold, it is critical that $a - s < 0$, that is,

$$s > a.$$