Problem 23

In each of Problems 21 through 24, find the Laplace transform of the given function.

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < \infty \end{cases}$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Split up the integral over the intervals that $f(t)$ is defined on.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^1 e^{-st}(t)\, dt + \int_1^\infty e^{-st}(1)\, dt \\
&= \int_0^1 te^{-st}\, dt + \int_1^\infty e^{-st}\, dt \\
&= \int_0^1 \left(-\frac{\partial}{\partial s} e^{-st}\right) dt - \frac{1}{s} e^{-st}\bigg|_1^\infty \\
&= -\frac{d}{ds} \int_0^1 e^{-st}\, dt - \frac{1}{s}(-e^{-s}) \\
&= -\frac{d}{ds}\left(-\frac{1}{s} e^{-st}\bigg|_0^1\right) + \frac{1}{s} e^{-s} \\
&= -\frac{d}{ds}\left[\frac{1}{s}(1 - e^{-s})\right] + \frac{1}{s} e^{-s} \\
&= -\left[-\frac{1}{s^2}(1 - e^{-s}) + \frac{1}{s} e^{-s}\right] + \frac{1}{s} e^{-s} \\
&= -\left(\frac{-1 + e^{-s} + se^{-s}}{s^2}\right) + \frac{1}{s} e^{-s} \\
&= \frac{1 - e^{-s} - se^{-s}}{s^2} + \frac{1}{s^2} se^{-s} \\
&= \frac{1 - e^{-s}}{s^2}
\end{aligned}$$

Below is a side-by-side comparison of the function $f(t)$ and its Laplace transform $F(s)$.