Problem 24

In each of Problems 21 through 24, find the Laplace transform of the given function.

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Split up the integral over the intervals that $f(t)$ is defined on.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^1 e^{-st}(t)\, dt + \int_1^2 e^{-st}(2-t)\, dt + \int_2^\infty e^{-st}(0)\, dt \\
&= \int_0^1 te^{-st}\, dt + 2\int_1^2 e^{-st}\, dt - \int_1^2 te^{-st}\, dt \\
&= \int_0^1 \left(-\frac{\partial}{\partial s} e^{-st}\right) dt - \frac{2}{s} e^{-st}\bigg|_1^2 - \int_1^2 \left(-\frac{\partial}{\partial s} e^{-st}\right) dt \\
&= -\frac{d}{ds}\int_0^1 e^{-st}\, dt - \frac{2}{s}(e^{-2s} - e^{-s}) + \frac{d}{ds}\int_1^2 e^{-st}\, dt \\
&= -\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_0^1\right) - \frac{2}{s}(e^{-2s} - e^{-s}) + \frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_1^2\right) \\
&= -\frac{d}{ds}\left[\frac{1}{s}(1 - e^{-s})\right] - \frac{2}{s}(e^{-2s} - e^{-s}) + \frac{d}{ds}\left[\frac{1}{s}(e^{-s} - e^{-2s})\right] \\
&= -\left[-\frac{1}{s^2}(1 - e^{-s}) + \frac{1}{s}(e^{-s})\right] - \frac{2}{s}(e^{-2s} - e^{-s}) + \left[-\frac{1}{s^2}(e^{-s} - e^{-2s}) + \frac{1}{s}(-e^{-s} + 2e^{-2s})\right] \\
&= \frac{1}{s^2}(1 - e^{-s}) - \frac{1}{s}e^{-s} - \frac{2}{s}(e^{-2s} - e^{-s}) - \frac{1}{s^2}(e^{-s} - e^{-2s}) + \frac{1}{s}(-e^{-s} + 2e^{-2s}) \\
&= \frac{1 - 2e^{-s} + e^{-2s}}{s^2} \\
&= \frac{(1 - e^{-s})^2}{s^2} \\
&= \left(\frac{1 - e^{-s}}{s}\right)^2
\end{aligned}$$

Below is a side-by-side comparison of the function $f(t)$ and its Laplace transform $F(s)$.