Problem 11

Recall that \( \cos bt = \frac{e^{ibt} + e^{-ibt}}{2} \) and that \( \sin bt = \frac{e^{ibt} - e^{-ibt}}{2i} \). In each of Problems 11 through 14, find the Laplace transform of the given function; \( a \) and \( b \) are real constants. Assume that the necessary elementary integration formulas extend to this case.

\[ f(t) = \sin bt \]

Solution

The Laplace transform of a function \( f(t) \) is defined as

\[
F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt.
\]

Find the Laplace transform of \( \sin bt \).

\[
\mathcal{L}\{\sin bt\} = \int_0^\infty e^{-st} \sin bt \, dt
= \int_0^\infty e^{-st} \frac{e^{ibt} - e^{-ibt}}{2i} \, dt
= \frac{1}{2i} \int_0^\infty e^{-st} (e^{ibt} - e^{-ibt}) \, dt
= \frac{1}{2i} \left( \int_0^\infty e^{ib-st} \, dt - \int_0^\infty e^{-ib-st} \, dt \right)
= \frac{1}{2i} \left[ \int_0^\infty e^{(ib-s)t} \, dt - \int_0^\infty e^{(-ib-s)t} \, dt \right]
= \frac{1}{2i} \left[ \frac{1}{ib-s} \left| e^{(ib-s)t} \right|_0^\infty - \frac{1}{-ib-s} \left| e^{(-ib-s)t} \right|_0^\infty \right]
= \frac{1}{2i} \left[ \frac{1}{ib-s} (-1) - \frac{1}{-ib-s} (-1) \right]
= \frac{1}{2i} \left( \frac{1}{s - ib} - \frac{1}{s + ib} \right)
= \frac{1}{2i} \left( \frac{s + ib - (s - ib)}{(s - ib)(s + ib)} \right)
= \frac{1}{2i} \left( \frac{-2ib}{s^2 - i^2b^2} \right)
= \frac{b}{s^2 + b^2}
\]

Note that for equation (1) to hold, it is critical that \( s > 0 \).