Problem 15

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; $n$ is a positive integer and $a$ is a real constant.

\[ f(t) = te^{at} \]

Solution

The Laplace transform of a function $f(t)$ is defined as

\[ F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt. \]

Find the Laplace transform of $te^{at}$.

\[
\mathcal{L}\{te^{at}\} = \int_0^\infty e^{-st} te^{at} \, dt \\
= \int_0^\infty te^{at-st} \, dt \\
= \int_0^\infty te^{(a-s)t} \, dt \\
= \int_0^\infty \frac{d}{dt} \left[ \frac{1}{a-s} e^{(a-s)t} \right] \, dt \\
= t \left[ \frac{1}{a-s} e^{(a-s)t} \right]_0^\infty - \int_0^\infty \frac{1}{a-s} e^{(a-s)t} \, dt \\
= -\frac{1}{a-s} \int_0^\infty e^{(a-s)t} \, dt \\
= -\frac{1}{(a-s)^2} e^{(a-s)t} \bigg|_0^\infty \\
= -\frac{1}{(a-s)^2} (-1) \\
= \frac{1}{(a-s)^2} \]  

Note that for equations (1) and (2) to hold, it is critical that $a - s < 0$, that is,

\[ s > a. \]