Problem 16

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; \( n \) is a positive integer and \( a \) is a real constant.

\[ f(t) = t \sin at \]

Solution

The Laplace transform of a function \( f(t) \) is defined as

\[ F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt. \]

Find the Laplace transform of \( t \sin at \).

\[
\mathcal{L}\{t \sin at\} = \int_0^\infty e^{-st} t \sin at \, dt
\]

\[
= \int_0^\infty e^{-st} t \frac{e^{iat} - e^{-iat}}{2i} \, dt
\]

\[
= \frac{1}{2i} \left( \int_0^\infty t e^{iat-st} \, dt - \int_0^\infty t e^{-iat-st} \, dt \right)
\]

\[
= \frac{1}{2i} \left[ \int_0^\infty t e^{(ia-s)t} \, dt - \int_0^\infty t e^{(-ia-s)t} \, dt \right]
\]

\[
= \frac{1}{2i} \left\{ \int_0^\infty \frac{d}{dt} \left[ \frac{1}{ia-s} e^{(ia-s)t} \right] \, dt - \int_0^\infty \frac{d}{dt} \left[ \frac{1}{-ia-s} e^{(-ia-s)t} \right] \, dt \right\}
\]

\[
= \frac{1}{2i} \left\{ t \left[ \frac{1}{ia-s} e^{(ia-s)t} \right]_0^\infty - \int_0^\infty \left[ \frac{1}{ia-s} e^{(ia-s)t} \right] \, dt - \left[ t \left[ \frac{1}{-ia-s} e^{(-ia-s)t} \right]_0^\infty \right] + \int_0^\infty \left[ \frac{1}{-ia-s} e^{(-ia-s)t} \right] \, dt \right\}
\]

\[
= \frac{1}{2i} \left\{ \frac{1}{ia-s} \int_0^\infty e^{(ia-s)t} \, dt + \frac{1}{-ia-s} \int_0^\infty e^{(-ia-s)t} \, dt \right\}
\]

\[
= \frac{1}{2i} \left[ \frac{1}{(ia-s)^2} e^{(ia-s)t} \bigg|_0^\infty - \frac{1}{(-ia-s)^2} e^{(-ia-s)t} \bigg|_0^\infty \right]
\]

\[
= \frac{1}{2i} \left[ \frac{1}{(ia-s)^2} (-1) + \frac{1}{(-ia-s)^2} (-1) \right]
\]

\[
= \frac{1}{2i} \left[ \frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2} \right]
\]

\[
= \frac{1}{2i} \left[ \frac{(s+ia)^2 - (s-ia)^2}{(s+ia)^2(s-ia)^2} \right]
\]

\[
= \frac{1}{2i} \left( \frac{4ias}{(s-ia)(s+ia)^2} \right)
\]

\[
= \frac{2as}{(s^2 + a^2)^2}
\]

Note that for equations (1) and (2) to hold, it is critical that \( s > 0 \).