Problem 20

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' + \omega^2 y = \cos 2t, \quad \omega^2 \neq 4; \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + \omega^2 y\} = \mathcal{L}\{\cos 2t\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \omega^2\mathcal{L}\{y\} = \mathcal{L}\{\cos 2t\}$$
$$[s^2Y(s) - sy(0) - y'(0)] + \omega^2 Y(s) = \frac{s}{s^2+4}$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$.

$$[s^2Y(s) - s] + \omega^2 Y(s) = \frac{s}{s^2+4}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2+\omega^2)Y(s) = \frac{s}{s^2+4} + s$$
$$Y(s) = \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}$$

Use partial fraction decomposition to split up the first term.

$$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{As+B}{s^2+\omega^2} + \frac{Cs+D}{s^2+4}$$

Multiply both sides by $(s^2+\omega^2)(s^2+4)$.

$$s = (As+B)(s^2+4) + (Cs+D)(s^2+\omega^2)$$

Choose four random values for s to get a system of equations for A, B, C, and D.

$$\begin{aligned}
s = 0: &\qquad 0 = 4B + \omega^2 D \\
s = 1: &\qquad 1 = 5A + 5B + (1+\omega^2)C + (1+\omega^2)D \\
s = 2: &\qquad 2 = 16A + 8B + 2(4+\omega^2)C + (4+\omega^2)D \\
s = 3: &\qquad 3 = 39A + 13B + 3(9+\omega^2)C + (9+\omega^2)D
\end{aligned}$$

Solving this system yields

$$A = -\frac{1}{\omega^2 - 4} \quad \text{and} \quad B = 0 \quad \text{and} \quad C = \frac{1}{\omega^2 - 4} \quad \text{and} \quad D = 0,$$

so $Y(s)$ becomes

$$Y(s) = -\frac{1}{\omega^2 - 4}\frac{s}{s^2 + \omega^2} + \frac{1}{\omega^2 - 4}\frac{s}{s^2 + 4} + \frac{s}{s^2 + \omega^2}.$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{-\frac{1}{\omega^2 - 4}\frac{s}{s^2 + \omega^2} + \frac{1}{\omega^2 - 4}\frac{s}{s^2 + 4} + \frac{s}{s^2 + \omega^2}\right\} \\
&= -\frac{1}{\omega^2 - 4}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\} + \frac{1}{\omega^2 - 4}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\} \\
&= -\frac{1}{\omega^2 - 4}\cos\omega t + \frac{1}{\omega^2 - 4}\cos 2t + \cos\omega t \\
&= \left(1 - \frac{1}{\omega^2 - 4}\right)\cos\omega t + \frac{1}{\omega^2 - 4}\cos 2t \\
&= \frac{\omega^2 - 5}{\omega^2 - 4}\cos\omega t + \frac{1}{\omega^2 - 4}\cos 2t
\end{aligned}$$