Problem 24

In each of Problems 24 through 27, find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3. You may wish to refer to Problems 21 through 24 in Section 6.1.

$$y'' + 4y = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < \infty; \end{cases} \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Let $f(t)$ represent the piecewise function on the right side.

$$y'' + 4y = f(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$$

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{f(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$, and $f(t)$.

$$[s^2Y(s) - s] + 4Y(s) = \int_0^\pi e^{-st}(1)\, dt + \int_\pi^\infty e^{-st}(0)\, dt$$

$$(s^2 + 4)Y(s) - s = \int_0^\pi e^{-st}\, dt$$

$$(s^2 + 4)Y(s) = s + \frac{1 - e^{-\pi s}}{s}$$

Divide both sides by s^2+4.

$$\begin{aligned}
Y(s) &= \frac{s}{s^2+4} + \frac{1-e^{-\pi s}}{s(s^2+4)} \\
&= \frac{s}{s^2+4} + \frac{1}{s(s^2+4)} - \frac{e^{-\pi s}}{s(s^2+4)} \\
&= \frac{s}{s^2+4} + \frac{1}{s(s^2+4)} - \frac{1}{s(s^2+4)}e^{-\pi s} \\
&= \frac{s}{s^2+4} + \frac{\frac{1}{4}}{s} - \frac{\frac{s}{4}}{s^2+4} - \frac{1}{s(s^2+4)}e^{-\pi s} \\
&= \frac{s}{s^2+4} + \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{s(s^2+4)}e^{-\pi s}
\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$. Note that $H(t)$ is the Heaviside function, which is defined to be 1 if $t>0$ and 0 if $t<0$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{s(s^2+4)}e^{-\pi s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}e^{-\pi s}\right\} \\
&= \cos 2t + \frac{1}{4} - \frac{1}{4}\cos 2t - \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-\pi)\right]H(t-\pi) \\
&= \frac{3}{4}\cos 2t + \frac{1}{4} - \frac{1}{4}[1-\cos 2(t-\pi)]H(t-\pi) \\
&= \frac{3}{4}\cos 2t + \frac{1}{4} - \frac{1}{4}(1-\cos 2t)H(t-\pi) \\
&= \frac{3}{4}\cos 2t + \frac{1}{4} - \frac{1}{4}(2\sin^2 t)H(t-\pi) \\
&= \frac{1}{4}[3\cos 2t + 1 - 2H(t-\pi)\sin^2 t]
\end{aligned}$$