Problem 28

The Laplace transforms of certain functions can be found conveniently from their Taylor series expansions.

(a) Using the Taylor series for $\sin t$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!},$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}, \qquad s > 1.$$

(b) Let

$$f(t) = \begin{cases} (\sin t)/t, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$\mathcal{L}\{f(t)\} = \arctan(1/s), \qquad s > 1.$$

(c) The Bessel function of the first kind of order zero, J_0, has the Taylor series (see Section 5.7)

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Assuming that the following Laplace transforms can be computed term by term, verify that

$$\mathcal{L}\{J_0(t)\} = (s^2+1)^{-1/2}, \qquad s > 1$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/(4s)}, \qquad s > 0.$$

Solution

Part (a)

Take the Laplace transform of both sides of the Taylor series expansion for $\sin t$.

$$\begin{aligned}
\mathcal{L}\{\sin t\} &= \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}\right\} \\
&= \sum_{n=0}^{\infty} \mathcal{L}\left\{\frac{(-1)^n t^{2n+1}}{(2n+1)!}\right\} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\mathcal{L}\{t^{2n+1}\} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{(2n+1)!}{s^{(2n+1)+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{2n+2}} \\
&= \frac{1}{s^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{s^{2n}} \\
&= \frac{1}{s^2}\sum_{n=0}^{\infty} \left(\frac{-1}{s^2}\right)^n \\
&= \frac{1}{s^2}\frac{1}{1-\left(\frac{-1}{s^2}\right)} \\
&= \frac{1}{s^2+1}
\end{aligned}$$

Part (b)

Divide the Taylor series expansion of $\sin t$ about $t = 0$ by t.

$$\begin{aligned}
f(t) &= \frac{\sin t}{t} \\
&= \frac{1}{t}\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}
\end{aligned}$$

Take the Laplace transform of both sides.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n+1)!}\right\} \\
&= \sum_{n=0}^{\infty}\mathcal{L}\left\{\frac{(-1)^n t^{2n}}{(2n+1)!}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\mathcal{L}\{t^{2n}\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{(2n)!}{s^{2n+1}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}s^{-2n-1} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(s^{-1})^{2n+1} \\
&= \arctan(s^{-1}) \\
&= \arctan(1/s)
\end{aligned}$$

Part (c)

Take the Laplace transform of both sides of the Taylor series expansion for $J_0(t)$.

$$\begin{aligned}
\mathcal{L}\{J_0(t)\} &= \mathcal{L}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}\right\} \\
&= \sum_{n=0}^{\infty}\mathcal{L}\left\{\frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\mathcal{L}\{t^{2n}\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\cdot\frac{(2n)!}{s^{2n+1}} \\
&= \frac{1}{s}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\cdot\frac{(2n)!}{s^{2n}} \\
&= \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}(s^{-2})^n \\
&= \frac{1}{s}\frac{1}{\sqrt{1+(s^{-2})}} \\
&= \frac{1}{\sqrt{s^2+1}}
\end{aligned}$$

Substitute $\sqrt{t}$ for t in the Taylor series expansion for $J_0(t)$.

$$\begin{aligned} J_0(\sqrt{t}) &= \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{t})^{2n}}{2^{2n}(n!)^2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(n!)^2} \end{aligned}$$

Take the Laplace transform of both sides.

$$\begin{aligned} \mathcal{L}\{J_0(\sqrt{t})\} &= \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{2^{2n}(n!)^2}\right\} \\ &= \sum_{n=0}^{\infty} \mathcal{L}\left\{\frac{(-1)^n t^n}{2^{2n}(n!)^2}\right\} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \mathcal{L}\{t^n\} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \cdot \frac{n!}{s^{n+1}} \\ &= \frac{1}{s} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!} \cdot \frac{1}{s^n} \\ &= \frac{1}{s} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}s^{-1}\right)^n}{n!} \\ &= \frac{1}{s} \exp\left(-\frac{1}{4}s^{-1}\right) \\ &= \frac{1}{s} e^{-1/(4s)} \end{aligned}$$