Problem 13

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

\[ y'' - 2y' + 2y = 0; \quad y(0) = 0, \quad y'(0) = 1 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t) \, dt. \]

Consequently, the first and second derivatives transform as follows.

\[
\mathcal{L}\left\{ \frac{dy}{dt} \right\} = sY(s) - y(0)
\]

\[
\mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} = s^2Y(s) - sy(0) - y'(0)
\]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{0\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0 \]

\[ [s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) = 0 \]

Plug in the initial conditions, \( y(0) = 0 \) and \( y'(0) = 1 \).

\[ [s^2Y(s) - 1] - 2[sY(s)] + 2Y(s) = 0 \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[ s^2Y(s) - 2sY(s) + 2Y(s) - 1 = 0 \]

\[ (s^2 - 2s + 2)Y(s) = 1 \]

\[ Y(s) = \frac{1}{s^2 - 2s + 2} \]

\[ = \frac{1}{s^2 - 2s + 1 + 1} \]

\[ = \frac{1}{(s - 1)^2 + 1} \]

Take the inverse Laplace transform of \( Y(s) \) now to recover \( y(t) \).

\[ y(t) = \mathcal{L}^{-1}\{Y(s)\} \]

\[ = \mathcal{L}^{-1}\left\{ \frac{1}{(s - 1)^2 + 1} \right\} \]

\[ = e^t \sin t \]