Problem 14

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

\[ y'' - 4y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 1 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the first and second derivatives transform as follows.

\[
\begin{align*}
\mathcal{L}\left\{ \frac{dy}{dt} \right\} & = sY(s) - y(0) \\
\mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} & = s^2Y(s) - sy(0) - y'(0)
\end{align*}
\]

Apply the Laplace transform to both sides of the ODE.

\[
\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{0\}
\]

Use the fact that the transform is a linear operator.

\[
\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0
\]

\[
[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 4Y(s) = 0
\]

Plug in the initial conditions, \( y(0) = 1 \) and \( y'(0) = 1 \).

\[
[s^2Y(s) - s - 1] - 4[sY(s) - 1] + 4Y(s) = 0
\]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[
s^2Y(s) - 4sY(s) + 4Y(s) - s - 1 + 4 = 0
\]

\[
(s^2 - 4s + 4)Y(s) = s - 3
\]

\[
Y(s) = \frac{s - 3}{s^2 - 4s + 4}
\]

\[
= \frac{s - 3}{(s - 2)^2}
\]

\[
= \frac{1}{s - 2} + \frac{-1}{(s - 2)^2}
\]

Take the inverse Laplace transform of \( Y(s) \) now to recover \( y(t) \).

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\}
\]

\[
= \mathcal{L}^{-1}\left\{ \frac{1}{s - 2} + \frac{-1}{(s - 2)^2} \right\}
\]

\[
= \mathcal{L}^{-1}\left\{ \frac{1}{s - 2} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{(s - 2)^2} \right\}
\]

\[
= e^{2t} - te^{2t}
\]

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