Problem 15

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

\[ y'' - 2y' + 4y = 0; \quad y(0) = 2, \quad y'(0) = 0 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the first and second derivatives transform as follows.

\[
\begin{align*}
\mathcal{L}\{\frac{dy}{dt}\} &= sY(s) - y(0) \\
\mathcal{L}\{\frac{d^2y}{dt^2}\} &= s^2Y(s) - sy(0) - y'(0)
\end{align*}
\]

Apply the Laplace transform to both sides of the ODE.

\[
\mathcal{L}\{y'' - 2y' + 4y\} = \mathcal{L}\{0\}
\]

Use the fact that the transform is a linear operator.

\[
\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0
\]

\[
[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 4Y(s) = 0
\]

Plug in the initial conditions, \( y(0) = 2 \) and \( y'(0) = 0 \).

\[
[s^2Y(s) - 2s] - 2[sY(s) - 2] + 4Y(s) = 0
\]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[
s^2Y(s) - 2sY(s) + 4Y(s) - 2s + 4 = 0
\]

\[
(s^2 - 2s + 4)Y(s) = 2s - 4
\]

\[
Y(s) = \frac{2s - 4}{s^2 - 2s + 4}
\]

\[
= \frac{2s - 4}{s^2 - 2s + 1 + 4 - 1}
\]

\[
= \frac{2s - 4}{(s - 1)^2 + 3}
\]

\[
= \frac{2s - 2 - 2}{(s - 1)^2 + 3}
\]

\[
= \frac{s - 1}{2(s - 1)^2 + 3}
\]

\[
= \frac{2}{(s - 1)^2 + 3} - \frac{2}{\sqrt{3}(s - 1)^2 + 3}
\]

\[
= \frac{s - 1}{(s - 1)^2 + 3} - \frac{2}{\sqrt{3}(s - 1)^2 + 3}
\]

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Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\} \\
= \mathcal{L}^{-1}\left\{\frac{2(s - 1)}{(s - 1)^2 + 3} - \frac{2\sqrt{3}}{\sqrt{3}(s - 1)^2 + 3}\right\} \\
= 2\mathcal{L}^{-1}\left\{\frac{s - 1}{(s - 1)^2 + 3}\right\} - \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s - 1)^2 + 3}\right\} \\
= 2e^t \cos \sqrt{3}t - \frac{2}{\sqrt{3}}e^t \sin \sqrt{3}t
\]