Problem 19

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

\[ y^{(4)} - 4y = 0; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the derivatives transform as follows.

\[
\begin{align*}
\mathcal{L}\left\{ \frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{ \frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{ \frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{ \frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{align*}
\]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\{y^{(4)} - 4y\} = \mathcal{L}\{0\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y^{(4)}\} - 4\mathcal{L}\{y\} = 0 \]

\[ [s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] - 4Y(s) = 0 \]

Plug in the initial conditions, \( y(0) = 1, \ y'(0) = 0, \ y''(0) = -2, \) and \( y'''(0) = 0. \)

\[ [s^4Y(s) - s^3 + 2s] - 4Y(s) = 0 \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y(s) \), the transformed solution.

\[
\begin{align*}
s^4Y(s) - 4Y(s) - s^3 + 2s &= 0 \\
(s^4 - 4)Y(s) &= s^3 - 2s \\
Y(s) &= \frac{s^3 - 2s}{s^4 - 4} \\
&= \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)} \\
&= \frac{s}{s^2 + 2}
\end{align*}
\]
Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2}\right\}$$

$$= \cos \sqrt{2}t$$