Problem 22

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

\[ y'' - 2y' + 2y = e^{-t}; \quad y(0) = 0, \quad y'(0) = 1 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the first and second derivatives transform as follows.

\[ \mathcal{L}\left\{ \frac{dy}{dt} \right\} = sY(s) - y(0) \]
\[ \mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} = s^2Y(s) - sy(0) - y'(0) \]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{e^{-t}\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\} \]

\[ [s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) = \frac{1}{s + 1} \]

Plug in the initial conditions, \( y(0) = 0 \) and \( y'(0) = 1 \).

\[ [s^2Y(s) - 1] - 2[sY(s)] + 2Y(s) = \frac{1}{s + 1} \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[ s^2Y(s) - 2sY(s) + 2Y(s) - 1 = \frac{1}{s + 1} \]

\[ (s^2 - 2s + 2)Y(s) = \frac{1}{s + 1} + 1 \]

\[ = \frac{s + 2}{s + 1} \]

Divide both sides by \( s^2 - 2s + 2 \).

\[ Y(s) = \frac{s + 2}{(s + 1)(s^2 - 2s + 2)} \]

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Use partial fraction decomposition to write $Y(s)$ in terms of known transforms.

$$\frac{s + 2}{(s + 1)(s^2 - 2s + 2)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 - 2s + 2}$$

Multiply both sides by $(s + 1)(s^2 - 2s + 2)$.

$$s + 2 = A(s^2 - 2s + 2) + (Bs + C)(s + 1)$$

Choose three random values for $s$ to get a system of equations for $A$, $B$, and $C$.

$$s = 0 : 2 = 2A + C$$
$$s = 1 : 3 = A + 2B + 2C$$
$$s = 2 : 4 = 2A + 6B + 3C$$

Solving this system yields

$$A = \frac{1}{5} \text{ and } B = -\frac{1}{5} \text{ and } C = \frac{8}{5},$$

so $Y(s)$ becomes

$$Y(s) = \frac{\frac{1}{5}}{s + 1} + \frac{-\frac{1}{5}s + \frac{8}{5}}{s^2 - 2s + 2}$$

$$= \frac{1}{5} \frac{s + 1}{s + 1} - \frac{1}{5} \frac{s - 8}{s^2 - 2s + 2}$$

$$= \frac{1}{5} \frac{s + 1}{s^2 - 2s + 1 + 2 - 1}$$

$$= \frac{1}{5} \frac{s + 1}{s - 8}$$

$$= \frac{1}{5} \frac{s + 1}{5(s - 1)^2 + 1}$$

$$= \frac{1}{5} \frac{s + 1}{s - 1 - 7}$$

$$= \frac{1}{5} \frac{s + 1}{5(s - 1)^2 + 1} + \frac{7}{5} \frac{1}{(s - 1)^2 + 1}.$$ 

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{5} \frac{s + 1}{s + 1} - \frac{1}{5} \frac{s - 1}{s - 1 - 8 + 1} + \frac{7}{5} \frac{1}{(s - 1)^2 + 1}\right\}$$

$$= \frac{1}{5} \mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} - \frac{1}{5} \mathcal{L}^{-1}\left\{\frac{s - 1}{(s - 1)^2 + 1}\right\} + \frac{7}{5} \mathcal{L}^{-1}\left\{\frac{1}{(s - 1)^2 + 1}\right\}$$

$$= \frac{1}{5} e^{-t} - \frac{1}{5} e^t \cos t + \frac{7}{5} e^t \sin t$$

$$= \frac{1}{5} [e^{-t} + e^t (7 \sin t - \cos t)]$$