Problem 36

Consider Bessel’s equation of order zero

\[ ty'' + y' + ty = 0. \]

Recall from Section 5.7 that \( t = 0 \) is a regular singular point for this equation, and therefore solutions may become unbounded as \( t \to 0 \). However, let us try to determine whether there are any solutions that remain finite at \( t = 0 \) and have finite derivatives there. Assuming that there is such a solution \( y = \phi(t) \), let \( Y(s) = \mathcal{L}\{\phi(t)\} \).

(a) Show that \( Y(s) \) satisfies

\[ (1 + s^2)Y'(s) + sY(s) = 0. \]

(b) Show that \( Y(s) = c(1 + s^2)^{-1/2} \), where \( c \) is an arbitrary constant.

(c) Writing \( (1 + s^2)^{-1/2} = s^{-1}(1 + s^{-2})^{-1/2} \), expanding in a binomial series valid for \( s > 1 \), and assuming that it is permissible to take the inverse transform term by term, show that

\[ y = c \sum_{n=0}^{\infty} \frac{(-1)^n2n}{2^{2n}(n!)^2} = cJ_0(t), \]

where \( J_0 \) is the Bessel function of the first kind of order zero. Note that \( J_0(0) = 1 \) and that \( J_0 \) has finite derivatives of all orders at \( t = 0 \). It was shown in Section 5.7 that the second solution of this equation becomes unbounded as \( t \to 0 \).