Problem 38

Suppose that

\[ g(t) = \int_0^t f(\tau) d\tau. \]

If \( G(s) \) and \( F(s) \) are the Laplace transforms of \( g(t) \) and \( f(t) \), respectively, show that

\[ G(s) = F(s)/s. \]

Solution

Take the Laplace transform of both sides of the definition for \( g(t) \).

\[
\mathcal{L}\{g(t)\} = \mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\} = \int_0^\infty e^{-st} \left[ \int_0^t f(\tau) d\tau \right] dt
\]

The current mode of integration in the \( t\tau \)-plane is shown below on the left.
Integrate over this domain as shown on the right to switch the order of integration.

\[ G(s) = \int_{0}^{\infty} \int_{\tau}^{\infty} e^{-st} f(\tau) \, dt \, d\tau \]

\[ = \int_{0}^{\infty} \left( -\frac{1}{s} e^{-st} \right) \bigg|_{\tau}^{\infty} f(\tau) \, d\tau \]

\[ = \int_{0}^{\infty} \left( \frac{1}{s} e^{-s\tau} \right) f(\tau) \, d\tau \]

\[ = \frac{1}{s} \int_{0}^{\infty} e^{-s\tau} f(\tau) \, d\tau \]

\[ = \frac{1}{s} \mathcal{L}\{f(t)\} \]

\[ = \frac{F(s)}{s} \]