Problem 39

In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that

\[ F(s) = \frac{P(s)}{Q(s)}, \]

where \( Q(s) \) is a polynomial of degree \( n \) with distinct zeros \( r_1, \ldots, r_n \), and \( P(s) \) is a polynomial of degree less than \( n \). In this case it is possible to show that \( P(s)/Q(s) \) has a partial fraction expansion of the form

\[ \frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \ldots + \frac{A_n}{s - r_n}, \tag{i} \]

where the coefficients \( A_1, \ldots, A_n \) must be determined.

(a) Show that

\[ A_k = \frac{P(r_k)}{Q'(r_k)}, \quad k = 1, \ldots, n. \tag{ii} \]

*Hint:* One way to do this is to multiply Eq. (i) by \( s - r_k \) and then to take the limit as \( s \to r_k \).

(b) Show that

\[ \mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \tag{iii} \]