Problem 15

In each of Problems 13 through 18, find the Laplace transform of the given function.

$$f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

For the provided function in particular, we have

$$\begin{aligned}
F(s) &= \int_0^\pi e^{-st}(0)\, dt + \int_\pi^{2\pi} e^{-st}(t-\pi)\, dt + \int_{2\pi}^\infty e^{-st}(0)\, dt \\
&= \int_\pi^{2\pi} te^{-st}\, dt - \pi \int_\pi^{2\pi} e^{-st}\, dt \\
&= \int_\pi^{2\pi} \left(-\frac{\partial}{\partial s} e^{-st}\right) dt - \pi \int_\pi^{2\pi} e^{-st}\, dt \\
&= -\frac{d}{ds} \int_\pi^{2\pi} e^{-st}\, dt - \pi \int_\pi^{2\pi} e^{-st}\, dt \\
&= -\frac{d}{ds} \left(-\frac{1}{s} e^{-st}\right)\bigg|_\pi^{2\pi} - \pi \left(-\frac{1}{s} e^{-st}\right)\bigg|_\pi^{2\pi} \\
&= -\frac{d}{ds} \left(\frac{e^{-\pi s} - e^{-2\pi s}}{s}\right) - \pi \left(\frac{e^{-\pi s} - e^{-2\pi s}}{s}\right) \\
&= -\left[\frac{(-\pi e^{-\pi s} + 2\pi e^{-2\pi s})s - 1(e^{-\pi s} - e^{-2\pi s})}{s^2}\right] - \pi \left(\frac{e^{-\pi s} - e^{-2\pi s}}{s}\right) \\
&= \frac{\pi s e^{-\pi s} - 2\pi s e^{-2\pi s} + e^{-\pi s} - e^{-2\pi s}}{s^2} - \frac{\pi e^{-\pi s} - \pi e^{-2\pi s}}{s} \\
&= \frac{\pi s e^{-\pi s} - 2\pi s e^{-2\pi s} + e^{-\pi s} - e^{-2\pi s} - \pi s e^{-\pi s} + \pi s e^{-2\pi s}}{s^2} \\
&= \frac{e^{-\pi s} - e^{-2\pi s} - \pi s e^{-2\pi s}}{s^2} \\
&= \frac{e^{-\pi s} - (1 + \pi s) e^{-2\pi s}}{s^2}.
\end{aligned}$$