Problem 18

In each of Problems 13 through 18, find the Laplace transform of the given function.

$$f(t) = t - u_1(t)(t-1), \qquad t \geq 0$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

For the provided function in particular,

$$\begin{aligned} f(t) &= t - u_1(t)(t-1) \\ &= t - (t-1)H(t-1), \end{aligned}$$

we have

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}[t - (t-1)H(t-1)]\, dt \\ &= \int_0^\infty te^{-st}\, dt - \int_0^\infty (t-1)e^{-st}H(t-1)\, dt \\ &= \int_0^\infty te^{-st}\, dt - \int_1^\infty (t-1)e^{-st}\, dt \\ &= \int_0^\infty te^{-st}\, dt - \int_1^\infty te^{-st}\, dt + \int_1^\infty e^{-st}\, dt \\ &= \int_0^\infty \left(-\frac{\partial}{\partial s} e^{-st}\right) dt - \int_1^\infty \left(-\frac{\partial}{\partial s} e^{-st}\right) dt + \int_1^\infty e^{-st}\, dt \\ &= -\frac{d}{ds}\int_0^\infty e^{-st}\, dt + \frac{d}{ds}\int_1^\infty e^{-st}\, dt + \int_1^\infty e^{-st}\, dt \\ &= -\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\right)\bigg|_0^\infty + \frac{d}{ds}\left(-\frac{1}{s}e^{-st}\right)\bigg|_1^\infty + \left(-\frac{1}{s}e^{-st}\right)\bigg|_1^\infty \\ &= -\frac{d}{ds}\left(\frac{1}{s}\right) + \frac{d}{ds}\left(\frac{1}{s}e^{-s}\right) + \left(\frac{1}{s}e^{-s}\right) \\ &= -\left(-\frac{1}{s^2}\right) + \left(-\frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right) + \left(\frac{1}{s}e^{-s}\right) \\ &= \frac{1}{s^2} - \frac{1}{s^2}e^{-s} \\ &= \frac{1 - e^{-s}}{s^2}. \end{aligned}$$