Problem 31

In each of Problems 30 through 33, find the Laplace transform of the given function. In Problem 33, assume that term-by-term integration of the infinite series is permissible.

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

Solution

The Laplace transform of a function $f(t)$ is defined to be

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Of course, for this integral to converge, it's necessary that $s > 0$. Split up the integral over the intervals that the given function is defined over.

$$\begin{aligned} F(s) &= \int_0^1 e^{-st}(1)\,dt + \int_1^2 e^{-st}(0)\,dt + \int_2^3 e^{-st}(1)\,dt + \int_3^\infty e^{-st}(0)\,dt \\ &= \int_0^1 e^{-st}\,dt + \int_2^3 e^{-st}\,dt \\ &= -\frac{1}{s}e^{-st}\bigg|_0^1 - \frac{1}{s}e^{-st}\bigg|_2^3 \\ &= \frac{1}{s} - \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} \\ &= \frac{1 - e^{-s} + e^{-2s} - e^{-3s}}{s} \end{aligned}$$