Problem 34

Let f satisfy $f(t+T) = f(t)$ for all $t \geq 0$ and for some fixed positive number T; f is said to be periodic with period T on $0 \leq t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

Solution

The Laplace transform of a function $f(t)$ is defined to be

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Since f is periodic over an interval of T, split up the integral over multiples of T.

$$F(s) = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

Since f is periodic, $f(t) = f(t + mT)$ for any integer m.

$$\begin{aligned} F(s) &= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t-T)\,dt + \int_{2T}^{3T} e^{-st} f(t-2T)\,dt + \cdots \\ &= \sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-st} f(t-nT)\,dt \end{aligned}$$

Make the substitution $x = t - nT$. Then $dx = dt$.

$$\begin{aligned} F(s) &= \sum_{n=0}^\infty \int_0^T e^{-s(x+nT)} f(x)\,dx \\ &= \sum_{n=0}^\infty \int_0^T e^{-sx} e^{-snT} f(x)\,dx \\ &= \left(\sum_{n=0}^\infty e^{-snT}\right) \int_0^T e^{-sx} f(x)\,dx \\ &= \left[\sum_{n=0}^\infty (e^{-sT})^n\right] \int_0^T e^{-sx} f(x)\,dx \\ &= \left(\frac{1}{1 - e^{-sT}}\right) \int_0^T e^{-sx} f(x)\,dx \end{aligned}$$

Therefore, replacing the dummy integration variable x with t,

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$