Problem 15

In each of Problems 13 through 18, find the Laplace transform of the given function.

\[ f(t) = \begin{cases} 
0, & t < \pi \\
(t - \pi), & \pi \leq t < 2\pi \\
0, & t \geq 2\pi 
\end{cases} \]

Solution

The Laplace transform of a function \( f(t) \) is defined here as

\[ F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) \, dt. \]

For the provided function in particular, we have

\[
F(s) = \int_{0}^{\pi} e^{-st} (0) \, dt + \int_{\pi}^{2\pi} e^{-st} (t - \pi) \, dt + \int_{2\pi}^{\infty} e^{-st} (0) \, dt
\]

\[
= \int_{\pi}^{2\pi} te^{-st} \, dt - \pi \int_{\pi}^{2\pi} e^{-st} \, dt
\]

\[
= \int_{\pi}^{2\pi} \left( -\frac{\partial}{\partial s} e^{-st} \right) \, dt - \pi \int_{\pi}^{2\pi} e^{-st} \, dt
\]

\[
= -\frac{d}{ds} \int_{\pi}^{2\pi} e^{-st} \, dt - \pi \int_{\pi}^{2\pi} e^{-st} \, dt
\]

\[
= -\frac{d}{ds} \left( -\frac{1}{s} e^{-st} \right)_{\pi}^{2\pi} - \pi \left( -\frac{1}{s} e^{-st} \right)_{\pi}^{2\pi}
\]

\[
= -\frac{d}{ds} \left( \frac{1}{s} e^{-st} \right)_{\pi}^{2\pi} - \pi \left( \frac{e^{-\pi s} - e^{-2\pi s}}{s} \right)
\]

\[
= -\left[ \frac{(-\pi e^{-\pi s} + 2\pi e^{-2\pi s}) s - 1(e^{-\pi s} - e^{-2\pi s})}{s^2} \right] - \pi \left( \frac{e^{-\pi s} - e^{-2\pi s}}{s} \right)
\]

\[
= \frac{\pi s e^{-\pi s} - 2\pi s e^{-2\pi s} + e^{-\pi s} - e^{-2\pi s}}{s^2} \quad - \quad \frac{\pi e^{-\pi s} - \pi e^{-2\pi s}}{s}
\]

\[
= \frac{\pi s e^{-\pi s} - 2\pi s e^{-2\pi s} + e^{-\pi s} - e^{-2\pi s}}{s^2} \quad - \quad \frac{\pi s e^{-\pi s} + \pi s e^{-2\pi s}}{s^2}
\]

\[
= \frac{e^{-\pi s} - e^{-2\pi s} - \pi s e^{-2\pi s}}{s^2}
\]

\[
= \frac{e^{-\pi s} - (1 + \pi s)e^{-2\pi s}}{s^2}
\]