Problem 38

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

\[ f(t) = \sin t, \quad 0 \leq t < \pi; \]
\[ f(t + \pi) = f(t). \]
See Figure 6.3.10.

Solution

For a function that repeats itself periodically every \( T \) units, the Laplace transform is

\[ \mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t) \, dt}{1 - e^{-sT}}. \]

In this problem the period is \( T = \pi \).

\[ \mathcal{L}\{f(t)\} = \frac{\int_0^\pi e^{-st}(\sin t) \, dt}{1 - e^{-\pi s}} \]

Use integration by parts twice to evaluate the integral in the numerator.

\[
\int_0^\pi e^{-st}(\sin t) \, dt = \int_0^\pi e^{-st} \frac{d}{dt}(\cos t) \, dt \\
= e^{-st}(\cos t) \bigg|_0^\pi - \int_0^\pi (-s)e^{-st}(\cos t) \, dt \\
= e^{-\pi s} + 1 - s \int_0^\pi e^{-st} \cos t \, dt \\
= e^{-\pi s} + 1 - s \int_0^\pi e^{-st} \frac{d}{dt}(\sin t) \, dt \\
= e^{-\pi s} + 1 - s \left[ e^{-st}(\sin t) \bigg|_0^\pi - \int_0^\pi (-s)e^{-st}(\sin t) \, dt \right] \\
= e^{-\pi s} + 1 - s^2 \int_0^\pi e^{-st}(\sin t) \, dt
\]
As a result,

\[(1 + s^2) \int_0^\pi e^{-st} \sin t \, dt = e^{-\pi s} + 1\]

\[\int_0^\pi e^{-st} \sin t \, dt = \frac{e^{-\pi s} + 1}{s^2 + 1},\]

and equation (1) becomes

\[\mathcal{L}\{f(t)\} = \frac{e^{-\pi s} + 1}{s^2 + 1} \cdot \frac{1}{1 - e^{-\pi s}} = \frac{e^{-\pi s} + 1}{(s^2 + 1)(1 - e^{-\pi s})}.\]