

## Problem 1

In each of Problems 1 through 13:

- Find the solution of the given initial value problem.
- Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \leq t < 3\pi \\ 0, & 3\pi \leq t < \infty \end{cases}$$

### Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function  $y(t)$  is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st} y(t) dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{f(t)\}$$

Use the fact that the transform is a linear operator.

$$\begin{aligned} \mathcal{L}\{y''\} + \mathcal{L}\{y\} &= \mathcal{L}\{f(t)\} \\ [s^2Y(s) - sy(0) - y'(0)] + Y(s) &= \int_0^{3\pi} e^{-st}(1) dt + \int_{3\pi}^{\infty} e^{-st}(0) dt \end{aligned}$$

Plug in the initial conditions,  $y(0) = 0$  and  $y'(0) = 1$ .

$$[s^2Y(s) - 1] + Y(s) = \int_0^{3\pi} e^{-st} dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for  $Y$ , the transformed solution.

$$\begin{aligned} (s^2 + 1)Y(s) - 1 &= \left(-\frac{1}{s}e^{-st}\right)\Big|_0^{3\pi} \\ (s^2 + 1)Y(s) &= \frac{1}{s} - \frac{1}{s}e^{-3\pi s} + 1 \\ Y(s) &= \frac{1}{s(s^2 + 1)} - \frac{1}{s(s^2 + 1)}e^{-3\pi s} + \frac{1}{s^2 + 1} \\ &= \left(\frac{1}{s} - \frac{s}{s^2 + 1}\right) - \left(\frac{1}{s} - \frac{s}{s^2 + 1}\right)e^{-3\pi s} + \frac{1}{s^2 + 1} \end{aligned}$$

Take the inverse Laplace transform of  $Y(s)$  now to get  $y(t)$ .

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
 &= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-3\pi s} + \frac{1}{s^2+1}\right\} \\
 &= (1 - \cos t) - [1 - \cos(t - 3\pi)]H(t - 3\pi) + \sin t \\
 &= 1 + \sin t - \cos t - [1 - \cos(t - \pi)]H(t - 3\pi) \\
 &= 1 + \sin t - \cos t - (1 + \cos t)H(t - 3\pi) \\
 &= 1 + \sin t - \cos t - (1 + \cos t)u_{3\pi}(t)
 \end{aligned}$$

Below is the graph of  $y(t)$  versus  $t$  superimposed on the graph of  $f(t)$  versus  $t$ .

