Problem 2

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + 2y' + 2y = h(t); \qquad y(0) = 0, \quad y'(0) = 1; \qquad h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \quad \text{and} \quad t \ge 2\pi \end{cases}$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{h(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{h(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2[Y(s)] = \int_\pi^{2\pi} e^{-st}(1)\, dt + \int_0^\pi e^{-st}(0)\, dt + \int_{2\pi}^\infty e^{-st}(0)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$[s^2Y(s) - 1] + 2[sY(s)] + 2[Y(s)] = \int_\pi^{2\pi} e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 2s + 2)Y(s) - 1 = \left(-\frac{1}{s}e^{-st}\right)\Bigg|_\pi^{2\pi}$$

$$(s^2 + 2s + 2)Y(s) = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s} + 1$$

$$Y(s) = \frac{1}{s(s^2 + 2s + 2)}e^{-\pi s} - \frac{1}{s(s^2 + 2s + 2)}e^{-2\pi s} + \frac{1}{s^2 + 2s + 2}$$

In order to write $Y(s)$ in terms of known transforms, use partial fraction decomposition.

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

Multiply both sides by $s(s^2+2s+2)$.

$$1 = A(s^2+2s+2) + (Bs+C)s$$

Plug in three random values of s to get a system of three equations for A, B, and C.

$$\begin{aligned} s=0: &\quad 1 = 2A \\ s=1: &\quad 1 = 5A+B+C \\ s=2: &\quad 1 = 10A+4B+2C \end{aligned}$$

Solving this system yields $A = 1/2$, $B = -1/2$, and $C = -1$.

$$Y(s) = \left(\frac{1/2}{s} + \frac{-\frac{1}{2}s-1}{s^2+2s+2}\right)e^{-\pi s} - \left(\frac{1/2}{s} + \frac{-\frac{1}{2}s-1}{s^2+2s+2}\right)e^{-2\pi s} + \frac{1}{s^2+2s+2}$$

Complete the square in the denominators.

$$\begin{aligned} Y(s) &= \left(\frac{1/2}{s} + \frac{-\frac{1}{2}s-1}{s^2+2s+1+2-1}\right)e^{-\pi s} - \left(\frac{1/2}{s} + \frac{-\frac{1}{2}s-1}{s^2+2s+1+2-1}\right)e^{-2\pi s} + \frac{1}{s^2+2s+1+2-1} \\ &= \left[\frac{1/2}{s} + \frac{-\frac{1}{2}s-1}{(s+1)^2+1}\right]e^{-\pi s} - \left[\frac{1/2}{s} + \frac{-\frac{1}{2}s-1}{(s+1)^2+1}\right]e^{-2\pi s} + \frac{1}{(s+1)^2+1} \end{aligned}$$

Make it so that $s+1$ appears in the numerators.

$$\begin{aligned} Y(s) &= \left[\frac{1/2}{s} + \frac{-\frac{1}{2}(s+1)-\frac{1}{2}}{(s+1)^2+1}\right]e^{-\pi s} - \left[\frac{1/2}{s} + \frac{-\frac{1}{2}(s+1)-\frac{1}{2}}{(s+1)^2+1}\right]e^{-2\pi s} + \frac{1}{(s+1)^2+1} \\ &= \left[\frac{1/2}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-\pi s} \\ &\quad - \left[\frac{1/2}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-2\pi s} + \frac{1}{(s+1)^2+1} \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\left[\frac{1/2}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-\pi s}\right. \\ &\qquad \left. - \left[\frac{1/2}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-2\pi s} + \frac{1}{(s+1)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\left[\frac{1/2}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-\pi s}\right\} \\ &\qquad - \mathcal{L}^{-1}\left\{\left[\frac{1/2}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-2\pi s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \end{aligned}$$

Evaluate the inverse Laplace transforms.

$$\begin{aligned} y(t) &= \left[\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi) - \frac{1}{2}e^{-(t-\pi)}\sin(t-\pi)\right]H(t-\pi) \\ &\qquad - \left[\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\cos(t-2\pi) - \frac{1}{2}e^{-(t-2\pi)}\sin(t-2\pi)\right]H(t-2\pi) + e^{-t}\sin t \\ &= \left(\frac{1}{2} + \frac{1}{2}e^{\pi-t}\cos t + \frac{1}{2}e^{\pi-t}\sin t\right)H(t-\pi) \\ &\qquad - \left(\frac{1}{2} - \frac{1}{2}e^{2\pi-t}\cos t - \frac{1}{2}e^{2\pi-t}\sin t\right)H(t-2\pi) + e^{-t}\sin t \\ &= \frac{1}{2}\left(1 + e^{\pi-t}\cos t + e^{\pi-t}\sin t\right)H(t-\pi) \\ &\qquad - \frac{1}{2}\left(1 - e^{2\pi-t}\cos t - e^{2\pi-t}\sin t\right)H(t-2\pi) + e^{-t}\sin t \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{2}\left(1 + e^{\pi-t}\cos t + e^{\pi-t}\sin t\right)u_\pi(t) - \frac{1}{2}\left(1 - e^{2\pi-t}\cos t - e^{2\pi-t}\sin t\right)u_{2\pi}(t) + e^{-t}\sin t.$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.