Problem 3

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi); \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\sin t - u_{2\pi}(t)\sin(t - 2\pi)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{u_{2\pi}(t)\sin(t - 2\pi)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] = \frac{1}{s^2+1} - \frac{1}{s^2+1}e^{-2\pi s}$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 4[Y(s)] = \frac{1}{s^2+1} - \frac{1}{s^2+1}e^{-2\pi s}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2+4)Y(s) = \frac{1}{s^2+1} - \frac{1}{s^2+1}e^{-2\pi s}$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} - \frac{1}{(s^2+1)(s^2+4)}e^{-2\pi s}$$

In order to write $Y(s)$ in terms of known transforms, use partial fraction decomposition.

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

Multiply both sides by $(s^2+1)(s^2+4)$.

$$1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

Plug in four random values of s to get a system of equations for A, B, C, and D.

$$\begin{aligned} s=0: &\quad 1 = 4B + D \\ s=1: &\quad 1 = 5A + 5B + 2C + 2D \\ s=2: &\quad 1 = 16A + 8B + 10C + 5D \\ s=3: &\quad 1 = 39A + 13B + 30C + 10D \end{aligned}$$

Solving this system yields $A = 0$, $B = 1/3$, $C = 0$, and $D = -1/3$.

$$\begin{aligned} Y(s) &= \frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4} - \left(\frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4}\right)e^{-2\pi s} \\ &= \frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4} - \left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right)e^{-2\pi s} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4} - \left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right)e^{-2\pi s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right)e^{-2\pi s}\right\} \\ &= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi)\right]H(t-2\pi) \\ &= \frac{1}{6}(2\sin t - \sin 2t) - \frac{1}{6}[2\sin(t-2\pi) - \sin 2(t-2\pi)]H(t-2\pi) \\ &= \frac{1}{6}(2\sin t - \sin 2t) - \frac{1}{6}(2\sin t - \sin 2t)u_{2\pi}(t) \\ &= \frac{1}{6}(2\sin t - 2\sin t\cos t) - \frac{1}{6}(2\sin t - 2\sin t\cos t)u_{2\pi}(t) \\ &= \frac{1}{3}\sin t(1-\cos t) - \frac{1}{3}\sin t(1-\cos t)u_{2\pi}(t) \\ &= \frac{1}{3}\sin t(1-\cos t)[1-u_{2\pi}(t)] \end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.