

Problem 4

In each of Problems 1 through 13:

- Find the solution of the given initial value problem.
- Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + 4y = \sin t + u_\pi(t) \sin(t - \pi); \quad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t) dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\sin t + u_\pi(t) \sin(t - \pi)\}$$

Use the fact that the transform is a linear operator.

$$\begin{aligned} \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} &= \mathcal{L}\{\sin t\} + \mathcal{L}\{u_\pi(t) \sin(t - \pi)\} \\ [s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] &= \frac{1}{s^2 + 1} + \frac{1}{s^2 + 1} e^{-\pi s} \end{aligned}$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 4[Y(s)] = \frac{1}{s^2 + 1} + \frac{1}{s^2 + 1} e^{-\pi s}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y , the transformed solution.

$$\begin{aligned} (s^2 + 4)Y(s) &= \frac{1}{s^2 + 1} + \frac{1}{s^2 + 1} e^{-\pi s} \\ Y(s) &= \frac{1}{(s^2 + 1)(s^2 + 4)} + \frac{1}{(s^2 + 1)(s^2 + 4)} e^{-\pi s} \end{aligned}$$

In order to write $Y(s)$ in terms of known transforms, use partial fraction decomposition.

$$\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}$$

Multiply both sides by $(s^2 + 1)(s^2 + 4)$.

$$1 = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1)$$

Plug in four random values of s to get a system of equations for A , B , C , and D .

$$s = 0 : 1 = 4B + D$$

$$s = 1 : 1 = 5A + 5B + 2C + 2D$$

$$s = 2 : 1 = 16A + 8B + 10C + 5D$$

$$s = 3 : 1 = 39A + 13B + 30C + 10D$$

Solving this system yields $A = 0$, $B = 1/3$, $C = 0$, and $D = -1/3$.

$$\begin{aligned} Y(s) &= \frac{\frac{1}{3}}{s^2 + 1} + \frac{-\frac{1}{3}}{s^2 + 4} + \left(\frac{\frac{1}{3}}{s^2 + 1} + \frac{-\frac{1}{3}}{s^2 + 4} \right) e^{-\pi s} \\ &= \frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} + \left(\frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} \right) e^{-\pi s} \end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{ \frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} + \left(\frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} \right) e^{-\pi s} \right\} \\ &= \mathcal{L}^{-1}\left\{ \frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} \right\} + \mathcal{L}^{-1}\left\{ \left(\frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} \right) e^{-\pi s} \right\} \\ &= \frac{1}{3} \sin t - \frac{1}{6} \sin 2t + \left[\frac{1}{3} \sin(t - \pi) - \frac{1}{6} \sin 2(t - \pi) \right] H(t - \pi) \\ &= \frac{1}{6} (2 \sin t - \sin 2t) + \frac{1}{6} [2 \sin(t - \pi) - \sin 2(t - \pi)] H(t - \pi) \\ &= \frac{1}{6} (2 \sin t - \sin 2t) + \frac{1}{6} (-2 \sin t - \sin 2t) u_\pi(t) \\ &= \frac{1}{6} (2 \sin t - 2 \sin t \cos t) - \frac{1}{6} (2 \sin t + 2 \sin t \cos t) u_\pi(t) \\ &= \frac{1}{3} \sin t (1 - \cos t) - \frac{1}{3} \sin t (1 + \cos t) u_\pi(t) \\ &= \frac{1}{3} \sin t [(1 - \cos t) - (1 + \cos t) u_\pi(t)] \end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t .

