Problem 8

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)(t - \pi/2); \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\left\{y'' + y' + \frac{5}{4}y\right\} = \mathcal{L}\{t - u_{\pi/2}(t)(t - \pi/2)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \frac{5}{4}\mathcal{L}\{y\} = \mathcal{L}\{t\} - \mathcal{L}\{u_{\pi/2}(t)(t - \pi/2)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + \frac{5}{4}[Y(s)] = \frac{1}{s^2} - \int_0^\infty e^{-st}[u_{\pi/2}(t)(t - \pi/2)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + [sY(s)] + \frac{5}{4}[Y(s)] = \frac{1}{s^2} - \int_{\pi/2}^\infty e^{-st}(t - \pi/2)\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$\begin{aligned}
\left(s^2 + s + \frac{5}{4}\right) Y(s) &= \frac{1}{s^2} - \int_{\pi/2}^\infty te^{-st}\, dt + \frac{\pi}{2}\int_{\pi/2}^\infty e^{-st}\, dt \\
&= \frac{1}{s^2} - \int_{\pi/2}^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right) dt + \frac{\pi}{2}\int_{\pi/2}^\infty e^{-st}\, dt \\
&= \frac{1}{s^2} + \frac{d}{ds}\int_{\pi/2}^\infty e^{-st}\, dt + \frac{\pi}{2}\int_{\pi/2}^\infty e^{-st}\, dt
\end{aligned}$$

Evaluate the integrals and simplify the right side.

$$\begin{aligned}\left(s^2+s+\frac{5}{4}\right)Y(s) &= \frac{1}{s^2}+\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_{\pi/2}^{\infty}\right)+\frac{\pi}{2}\left(-\frac{1}{s}e^{-st}\bigg|_{\pi/2}^{\infty}\right)\\ &= \frac{1}{s^2}+\frac{d}{ds}\left(\frac{1}{s}e^{-\pi s/2}\right)+\frac{\pi}{2}\left(\frac{1}{s}e^{-\pi s/2}\right)\\ &= \frac{1}{s^2}+\left(-\frac{1}{s^2}e^{-\pi s/2}-\frac{\pi}{2}\frac{1}{s}e^{-\pi s/2}\right)+\frac{\pi}{2}\frac{1}{s}e^{-\pi s/2}\\ &= \frac{1}{s^2}-\frac{1}{s^2}e^{-\pi s/2}\end{aligned}$$

Divide both sides by $s^2+s+\frac{5}{4}$.

$$Y(s)=\frac{1}{s^2(s^2+s+\frac{5}{4})}-\frac{1}{s^2(s^2+s+\frac{5}{4})}e^{-\pi s/2}$$

Use partial fraction decomposition now.

$$\frac{1}{s^2(s^2+s+\frac{5}{4})}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+s+\frac{5}{4}}$$

Multiply both sides by $s^2(s^2+s+\frac{5}{4})$.

$$1=As(s^2+s+\frac{5}{4})+B(s^2+s+\frac{5}{4})+s^2(Cs+D)$$

Plug in four random values of s to obtain a system of four equations for A, B, C, and D.

$$\begin{aligned} s=0: &\quad 1=\frac{5}{4}B\\ s=1: &\quad 1=\frac{13}{4}A+\frac{13}{4}B+C+D\\ s=2: &\quad 1=\frac{29}{2}A+\frac{29}{4}B+8C+4D\\ s=3: &\quad 1=\frac{159}{4}A+\frac{53}{4}B+27C+9D\end{aligned}$$

Solving this system yields $A=-16/25$, $B=4/5$, $C=16/25$, and $D=-4/25$.

$$Y(s)=\left(\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}s-\frac{4}{25}}{s^2+s+\frac{5}{4}}\right)-\left(\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}s-\frac{4}{25}}{s^2+s+\frac{5}{4}}\right)e^{-\pi s/2}$$

Complete the square in the denominators.

$$\begin{aligned}Y(s)&=\left(\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}s-\frac{4}{25}}{s^2+s+\frac{1}{4}+\frac{5}{4}-\frac{1}{4}}\right)-\left(\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}s-\frac{4}{25}}{s^2+s+\frac{1}{4}+\frac{5}{4}-\frac{1}{4}}\right)e^{-\pi s/2}\\ &=\left[\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}s-\frac{4}{25}}{\left(s+\frac{1}{2}\right)^2+1}\right]-\left[\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}s-\frac{4}{25}}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s/2}\end{aligned}$$

Make it so that $s+\frac{1}{2}$ appears in the numerators.

$$\begin{aligned}
Y(s) &= \left[\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}\left(s+\frac{1}{2}\right)-\frac{16}{50}-\frac{4}{25}}{\left(s+\frac{1}{2}\right)^2+1}\right]-\left[\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}\left(s+\frac{1}{2}\right)-\frac{16}{50}-\frac{4}{25}}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s/2} \\
&= \left[\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}\left(s+\frac{1}{2}\right)-\frac{12}{25}}{\left(s+\frac{1}{2}\right)^2+1}\right]-\left[\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}\left(s+\frac{1}{2}\right)-\frac{12}{25}}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s/2}
\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}\left(s+\frac{1}{2}\right)-\frac{12}{25}}{\left(s+\frac{1}{2}\right)^2+1}\right\}-\mathcal{L}^{-1}\left\{\left[\frac{-\frac{16}{25}}{s}+\frac{\frac{4}{5}}{s^2}+\frac{\frac{16}{25}\left(s+\frac{1}{2}\right)-\frac{12}{25}}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s/2}\right\} \\
&= -\frac{16}{25}+\frac{4}{5}t+\frac{16}{25}e^{-t/2}\cos t-\frac{12}{25}e^{-t/2}\sin t \\
&\qquad -\left[-\frac{16}{25}+\frac{4}{5}\left(t-\frac{\pi}{2}\right)+\frac{16}{25}e^{-\left(t-\frac{\pi}{2}\right)/2}\cos\left(t-\frac{\pi}{2}\right)-\frac{12}{25}e^{-\left(t-\frac{\pi}{2}\right)/2}\sin\left(t-\frac{\pi}{2}\right)\right]H\left(t-\frac{\pi}{2}\right) \\
&= \frac{4}{25}(-4+5t+4e^{-t/2}\cos t-3e^{-t/2}\sin t) \\
&\qquad -\left[-\frac{2}{25}(5\pi+8)+\frac{4}{5}t+\frac{16}{25}e^{(\pi-2t)/4}\sin t+\frac{12}{25}e^{(\pi-2t)/4}\cos t\right]H\left(t-\frac{\pi}{2}\right) \\
&= \frac{4}{25}[-4+5t+e^{-t/2}(4\cos t-3\sin t)]+\frac{2}{25}\left[(5\pi+8)-10t-2e^{(\pi-2t)/4}(4\sin t+3\cos t)\right]H\left(t-\frac{\pi}{2}\right) \\
&= \frac{4}{25}[-4+5t+e^{-t/2}(4\cos t-3\sin t)]+\frac{2}{25}\left[(5\pi+8)-10t-2e^{(\pi-2t)/4}(4\sin t+3\cos t)\right]u_{\pi/2}(t)
\end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $f(t)$ versus t.