Problem 10

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

$$y'' + y' + \frac{5}{4}y = g(t); \qquad y(0) = 0, \quad y'(0) = 0; \qquad g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\left\{y'' + y' + \frac{5}{4}y\right\} = \mathcal{L}\{g(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \frac{5}{4}\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + \frac{5}{4}[Y(s)] = \int_0^\pi e^{-st}(\sin t)\, dt + \int_\pi^\infty e^{-st}(0)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + [sY(s)] + \frac{5}{4}[Y(s)] = \int_0^\pi e^{-st} \sin t\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$\left(s^2 + s + \frac{5}{4}\right) Y(s) = \frac{1}{s^2+1} + \frac{1}{s^2+1}e^{-\pi s}$$

Solve for $Y(s)$ and write the right side in terms of known transforms.

$$Y(s) = \frac{1}{(s^2+1)\left(s^2+s+\frac{5}{4}\right)} + \frac{1}{(s^2+1)\left(s^2+s+\frac{5}{4}\right)}e^{-\pi s}$$

Use partial fraction decomposition.

$$\frac{1}{(s^2+1)\left(s^2+s+\frac{5}{4}\right)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+s+\frac{5}{4}}$$

Multiply both sides by $(s^2+1)\left(s^2+s+\frac{5}{4}\right)$.

$$1 = (As+B)\left(s^2+s+\frac{5}{4}\right) + (Cs+D)(s^2+1)$$

Plug in four random values for s to obtain a system of four equations for A, B, C, and D.

$$\begin{aligned} s=0: &\quad 1 = \frac{5}{4}B + D \\ s=1: &\quad 1 = \frac{13}{4}A + \frac{13}{4}B + 2C + 2D \\ s=2: &\quad 1 = \frac{29}{2}A + \frac{29}{4}B + 10C + 5D \\ s=3: &\quad 1 = \frac{159}{4}A + \frac{53}{4}B + 30C + 10D \end{aligned}$$

Solving this system yields $A = -16/17$, $B = 4/17$, $C = 16/17$, and $D = 12/17$.

$$Y(s) = \left(\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}s+\frac{12}{17}}{s^2+s+\frac{5}{4}}\right) + \left(\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}s+\frac{12}{17}}{s^2+s+\frac{5}{4}}\right)e^{-\pi s}$$

Complete the square in the denominators.

$$\begin{aligned} Y(s) &= \left(\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}s+\frac{12}{17}}{s^2+s+\frac{1}{4}+\frac{5}{4}-\frac{1}{4}}\right) + \left(\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}s+\frac{12}{17}}{s^2+s+\frac{1}{4}+\frac{5}{4}-\frac{1}{4}}\right)e^{-\pi s} \\ &= \left[\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}s+\frac{12}{17}}{\left(s+\frac{1}{2}\right)^2+1}\right] + \left[\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}s+\frac{12}{17}}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s} \end{aligned}$$

Make it so that $s+\frac{1}{2}$ is in the numerators.

$$\begin{aligned} Y(s) &= \left[\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}\left(s+\frac{1}{2}\right)-\frac{16}{34}+\frac{12}{17}}{\left(s+\frac{1}{2}\right)^2+1}\right] + \left[\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}\left(s+\frac{1}{2}\right)-\frac{16}{34}+\frac{12}{17}}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s} \\ &= \left[\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}\left(s+\frac{1}{2}\right)+\frac{4}{17}}{\left(s+\frac{1}{2}\right)^2+1}\right] + \left[\frac{-\frac{16}{17}s+\frac{4}{17}}{s^2+1} + \frac{\frac{16}{17}\left(s+\frac{1}{2}\right)+\frac{4}{17}}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s} \\ &= \left[-\frac{16}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1} + \frac{16}{17}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} + \frac{4}{17}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\right] \\ &\qquad + \left[-\frac{16}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1} + \frac{16}{17}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} + \frac{4}{17}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s} \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{-\frac{16}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1} + \frac{16}{17}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} + \frac{4}{17}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\right. \\ &\qquad \left. + \left[-\frac{16}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1} + \frac{16}{17}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} + \frac{4}{17}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s}\right\} \end{aligned}$$

Use the fact that the inverse Laplace transform is a linear operator.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{-\frac{16}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1} + \frac{16}{17}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} + \frac{4}{17}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\right\} \\
&\qquad + \mathcal{L}^{-1}\left\{\left[-\frac{16}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1} + \frac{16}{17}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} + \frac{4}{17}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}\right]e^{-\pi s}\right\} \\
&= -\frac{16}{17}\cos t + \frac{4}{17}\sin t + \frac{16}{17}e^{-t/2}\cos t + \frac{4}{17}e^{-t/2}\sin t \\
&\qquad + \left[-\frac{16}{17}\cos(t-\pi) + \frac{4}{17}\sin(t-\pi) + \frac{16}{17}e^{-(t-\pi)/2}\cos(t-\pi) + \frac{4}{17}e^{-(t-\pi)/2}\sin(t-\pi)\right]H(t-\pi) \\
&= \frac{4}{17}[-4\cos t + \sin t + e^{-t/2}(4\cos t + \sin t)] + \frac{4}{17}\left[4\cos t - \sin t - e^{(\pi-t)/2}(4\cos t + \sin t)\right]H(t-\pi) \\
&= \frac{4}{17}\left[4\cos t - \sin t - e^{\pi/2}e^{-t/2}(4\cos t + \sin t)\right]H(t-\pi) - \frac{4}{17}[4\cos t - \sin t - e^{-t/2}(4\cos t + \sin t)] \\
&= \frac{4}{17}(4\cos t - \sin t)[H(t-\pi) - 1] - \frac{4}{17}e^{-t/2}(4\cos t + \sin t)[e^{\pi/2}H(t-\pi) - 1]
\end{aligned}$$

Below is the graph of $y(t)$ versus t superimposed on the graph of $g(t)$ versus t.