

Problem 20

Consider the initial value problem

$$y'' + 0.1y' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $f(t)$ is the same as in Problem 19.

- Plot the graph of the solution. Use a large enough value of n and a long enough t -interval so that the transient part of the solution has become negligible and the steady state is clearly shown.
- Estimate the amplitude and frequency of the steady state part of the solution.
- Compare the results of part (b) with those from Section 3.8 for a sinusoidally forced oscillator.

Solution

Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function $y(t)$ is defined to be

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st} y(t) dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Substitute the function for $f(t)$ and take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + 0.1y' + y\} = \mathcal{L}\left\{u_0(t) + 2 \sum_{k=1}^n (-1)^k u_{k\pi}(t)\right\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 0.1\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{u_0(t)\} + 2 \sum_{k=1}^n (-1)^k \mathcal{L}\{u_{k\pi}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 0.1[sY(s) - y(0)] + Y(s) = \int_0^{\infty} e^{-st} u_0(t) dt + 2 \sum_{k=1}^n (-1)^k \int_0^{\infty} e^{-st} u_{k\pi}(t) dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 0.1[sY(s)] + Y(s) = \int_0^{\infty} e^{-st} dt + 2 \sum_{k=1}^n (-1)^k \int_{k\pi}^{\infty} e^{-st} dt$$

$$\left(s^2 + \frac{1}{10}s + 1\right) Y(s) = \frac{1}{s} + 2 \sum_{k=1}^n (-1)^k \left(\frac{e^{-k\pi s}}{s}\right)$$

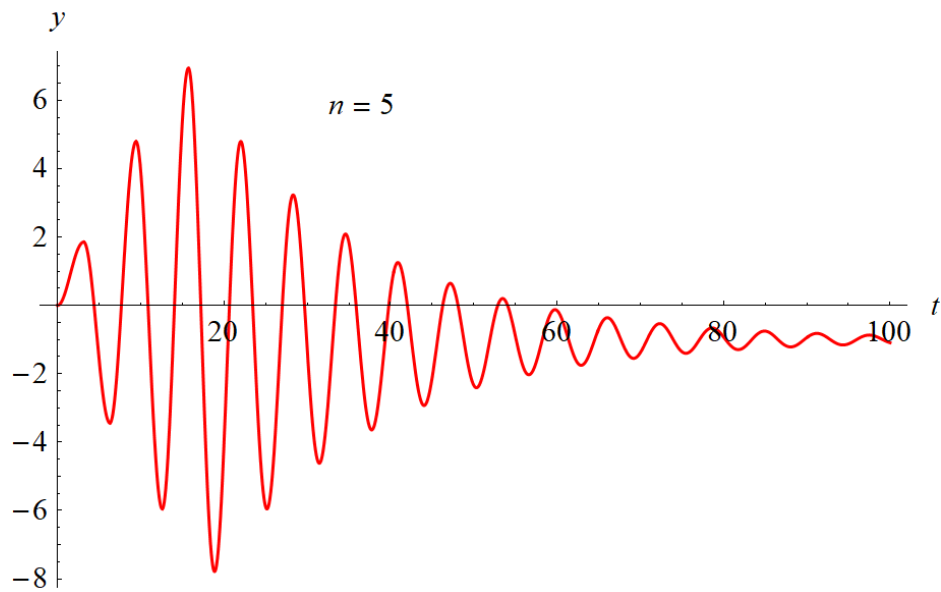
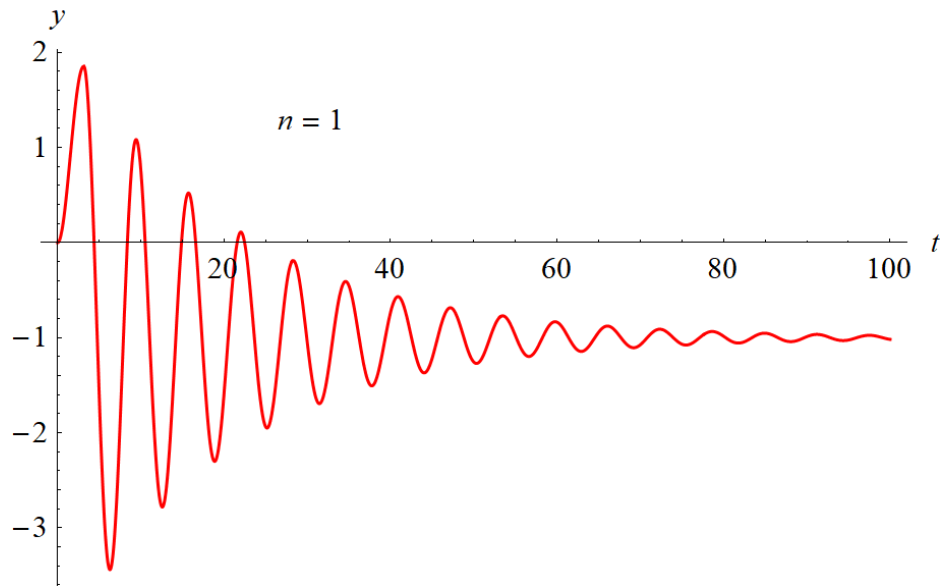
Solve for $Y(s)$ and then write it in terms of known transforms by using partial fraction decomposition and then completing the square.

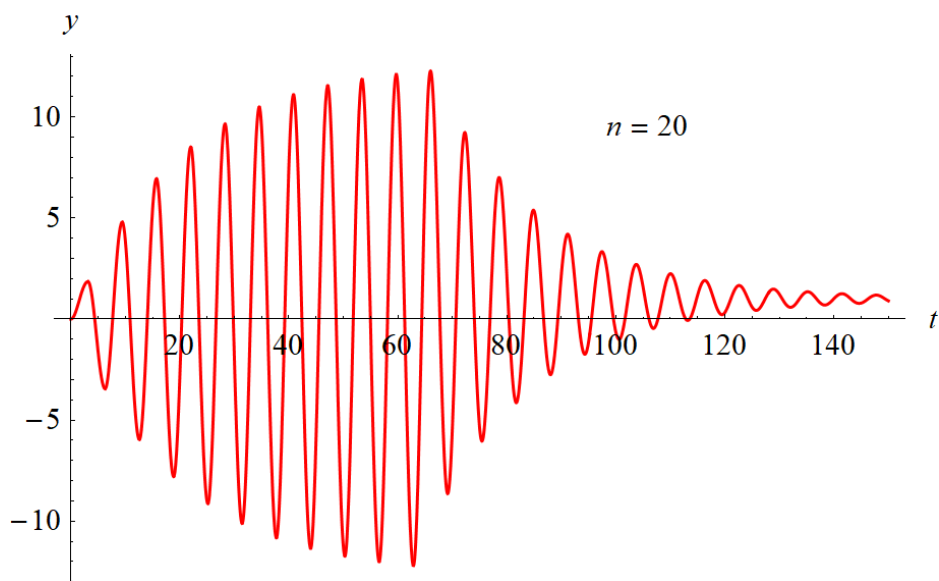
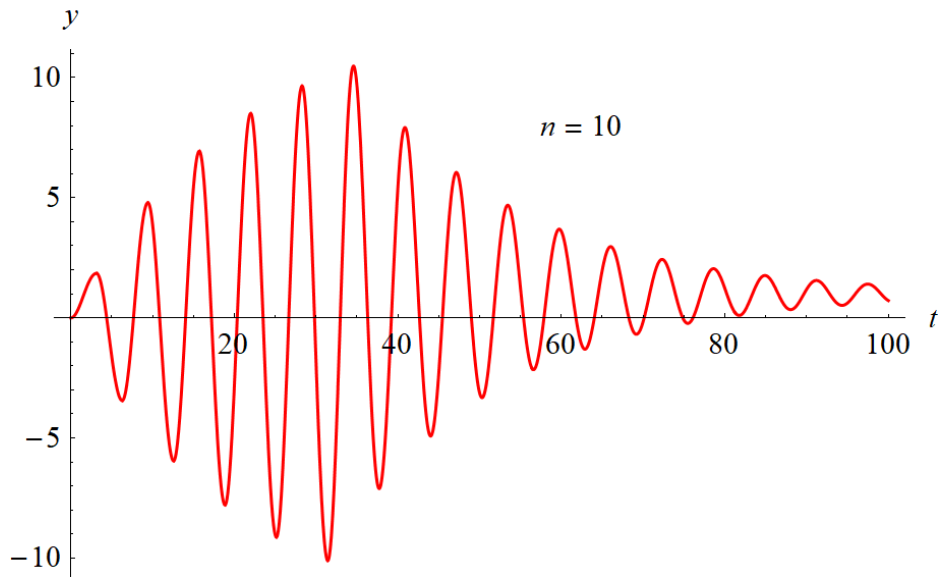
$$\begin{aligned}
 Y(s) &= \frac{1}{s(s^2 + \frac{1}{10}s + 1)} + 2 \sum_{k=1}^n (-1)^k \left[\frac{1}{s(s^2 + \frac{1}{10}s + 1)} \right] e^{-k\pi s} \\
 &= \left(\frac{1}{s} + \frac{-s - \frac{1}{10}}{s^2 + \frac{1}{10}s + 1} \right) + 2 \sum_{k=1}^n (-1)^k \left(\frac{1}{s} + \frac{-s - \frac{1}{10}}{s^2 + \frac{1}{10}s + 1} \right) e^{-k\pi s} \\
 &= \left(\frac{1}{s} - \frac{s + \frac{1}{10}}{s^2 + \frac{1}{10}s + \frac{1}{400} + 1 - \frac{1}{400}} \right) + 2 \sum_{k=1}^n (-1)^k \left(\frac{1}{s} - \frac{s + \frac{1}{10}}{s^2 + \frac{1}{10}s + \frac{1}{400} + 1 - \frac{1}{400}} \right) e^{-k\pi s} \\
 &= \left[\frac{1}{s} - \frac{s + \frac{1}{10}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] + 2 \sum_{k=1}^n (-1)^k \left[\frac{1}{s} - \frac{s + \frac{1}{10}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] e^{-k\pi s} \\
 &= \left[\frac{1}{s} - \frac{(s + \frac{1}{20}) + \frac{1}{10} - \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] + 2 \sum_{k=1}^n (-1)^k \left[\frac{1}{s} - \frac{(s + \frac{1}{20}) + \frac{1}{10} - \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] e^{-k\pi s} \\
 &= \left[\frac{1}{s} - \frac{(s + \frac{1}{20}) + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] + 2 \sum_{k=1}^n (-1)^k \left[\frac{1}{s} - \frac{(s + \frac{1}{20}) + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] e^{-k\pi s} \\
 &= \left[\frac{1}{s} - \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} - \frac{\frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] + 2 \sum_{k=1}^n (-1)^k \left[\frac{1}{s} - \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} - \frac{\frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] e^{-k\pi s} \\
 &= \left[\frac{1}{s} - \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} - \frac{1}{\sqrt{399}} \frac{\frac{\sqrt{399}}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] \\
 &\quad + 2 \sum_{k=1}^n (-1)^k \left[\frac{1}{s} - \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} - \frac{1}{\sqrt{399}} \frac{\frac{\sqrt{399}}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] e^{-k\pi s}
 \end{aligned}$$

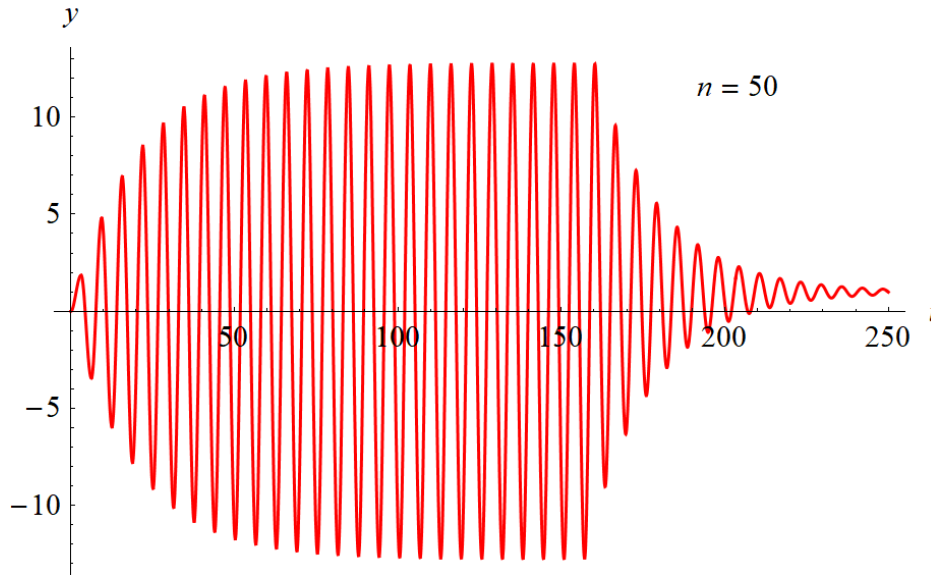
Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
 y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
 &= \mathcal{L}^{-1} \left\{ \left[\frac{1}{s} - \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} - \frac{1}{\sqrt{399}} \frac{\frac{\sqrt{399}}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] \right. \\
 &\quad \left. + 2 \sum_{k=1}^n (-1)^k \left[\frac{1}{s} - \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} - \frac{1}{\sqrt{399}} \frac{\frac{\sqrt{399}}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] e^{-k\pi s} \right\} \\
 &= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} - \frac{1}{\sqrt{399}} \frac{\frac{\sqrt{399}}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right\} \\
 &\quad + 2 \sum_{k=1}^n (-1)^k \mathcal{L}^{-1} \left\{ \left[\frac{1}{s} - \frac{s + \frac{1}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} - \frac{1}{\sqrt{399}} \frac{\frac{\sqrt{399}}{20}}{(s + \frac{1}{20})^2 + \frac{399}{400}} \right] e^{-k\pi s} \right\} \\
 &= 1 - e^{-t/20} \cos \frac{\sqrt{399}}{20} t - \frac{1}{\sqrt{399}} e^{-t/20} \sin \frac{\sqrt{399}}{20} t \\
 &\quad + 2 \sum_{k=1}^n (-1)^k \left\{ 1 - e^{-(t-k\pi)/20} \cos \left[\frac{\sqrt{399}}{20} (t - k\pi) \right] - \frac{1}{\sqrt{399}} e^{-(t-k\pi)/20} \sin \left[\frac{\sqrt{399}}{20} (t - k\pi) \right] \right\} H(t - k\pi)
 \end{aligned}$$

Below are several graphs of $y(t)$ versus t for various values of n .



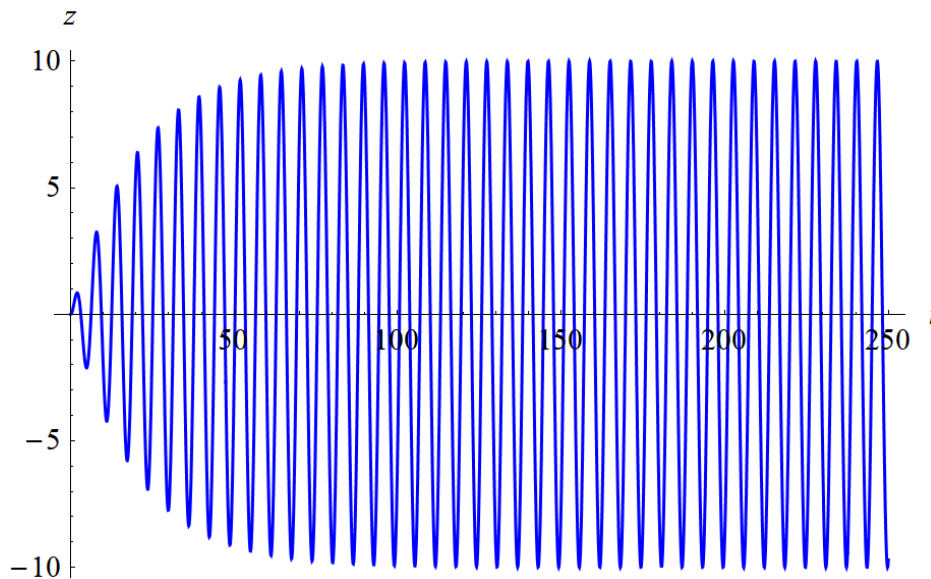




The amplitude of the steady part of the solution is about 12.6, and the period is $T = 2\pi$, which means the angular frequency is $\omega = 2\pi/T = 1$. The corresponding problem with a sinusoidal forcing term was considered in Section 3.8. Note that $\cos t$ most closely resembles $f(t)$.

$$z'' + 0.1z' + z = \cos t, \quad z(0) = 0, \quad z'(0) = 0$$

The solution to this ODE is plotted below.



The amplitude of the steady part of the solution here is about 9.9, and the period is $T = 2\pi$, which means the angular frequency is $\omega = 2\pi/T = 1$. This difference in amplitude is due to the fact that the cosine forcing varies in strength with time, whereas $f(t)$ delivers pulses at maximum strength.