Problem 1

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

\[ y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \leq t < 3\pi \\ 0, & 3\pi \leq t < \infty \end{cases} \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the first and second derivatives transform as follows.

\[ \mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) \]

\[ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0) \]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\{y'' + y\} = \mathcal{L}\{f(t)\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\} \]

\[ [s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^{3\pi} e^{-st}(1)\,dt + \int_{3\pi}^\infty e^{-st}(0)\,dt \]

Plug in the initial conditions, \( y(0) = 0 \) and \( y'(0) = 1 \).

\[ [s^2Y(s) - 1] + Y(s) = \int_0^{3\pi} e^{-st}\,dt \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[ (s^2 + 1)Y(s) - 1 = \left( -\frac{1}{s}e^{-st} \right)\bigg|_{t=3\pi}^{t=0} \]

\[ (s^2 + 1)Y(s) = \frac{1}{s} - \frac{1}{s}e^{-3\pi s} + 1 \]

\[ Y(s) = \frac{1}{s(s^2 + 1)} - \frac{1}{s(s^2 + 1)}e^{-3\pi s} + \frac{1}{s^2 + 1} \]

\[ = \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) - \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right)e^{-3\pi s} + \frac{1}{s^2 + 1} \]
Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{ \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) - \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-3\pi s} + \frac{1}{s^2 + 1} \right\}$$

$$= (1 - \cos t) - \left[ 1 - \cos(t - 3\pi) \right] H(t - 3\pi) + \sin t$$

$$= 1 + \sin t - \cos t - (1 + \cos t) H(t - 3\pi)$$

$$= 1 + \sin t - \cos t - (1 + \cos t)u_{3\pi}(t)$$

Below is the graph of $y(t)$ versus $t$ superimposed on the graph of $f(t)$ versus $t$. 

![Graph of y(t) versus t superimposed on the graph of f(t) versus t.](www.stemjock.com)