Problem 13

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

\[ y^{(4)} + 5y'' + 4y = 1 - u_\pi(t); \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the derivatives transform as follows.

\[
\begin{align*}
\mathcal{L}\left\{ \frac{dy}{dt} \right\} & = sY(s) - y(0) \\
\mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} & = s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{ \frac{d^3y}{dt^3} \right\} & = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{ \frac{d^4y}{dt^4} \right\} & = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{align*}
\]

Apply the Laplace transform to both sides of the ODE.

\[
\mathcal{L}\{y^{(4)} + 5y'' + 4y\} = \mathcal{L}\{1 - u_\pi(t)\}
\]

Use the fact that the transform is a linear operator.

\[
\mathcal{L}\{y^{(4)}\} + 5\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{u_\pi(t)\}
\]

\[
\begin{align*}
[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] & + 5[s^2Y(s) - sy(0) - y'(0)] \\
& + 4[Y(s)] = \int_0^\infty e^{-st}(1)\,dt - \int_0^\infty e^{-st}[u_\pi(t)]\,dt
\end{align*}
\]

Plug in the initial conditions, \( y(0) = 0, \ y'(0) = 0, \ y''(0) = 0, \) and \( y'''(0) = 0. \)

\[
\begin{align*}
[s^4Y(s)] & + 5[s^2Y(s)] + 4[Y(s)] = \int_0^\infty e^{-st}\,dt - \int_0^\infty e^{-st}\,dt
\end{align*}
\]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y, \) the transformed solution.

\[
(s^4 + 5s^2 + 4)Y(s) = \left( \frac{-1}{s}e^{-st} \right)\bigg|_0^\infty - \left( \frac{-1}{s}e^{-st} \right)\bigg|_\pi
\]

\[
= \frac{1}{s} - \frac{1}{s}e^{-\pi s}
\]

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Solve for \( Y(s) \) and write the right side in terms of known transforms.

\[
Y(s) = \frac{1}{s(s^2 + 1)(s^2 + 4)} - \frac{1}{s(s^2 + 1)(s^2 + 4)} e^{-\pi s}
= \frac{1}{s(s^2 + 1)(s^2 + 4)} - \frac{1}{s(s^2 + 1)(s^2 + 4)} e^{-\pi s}
\]

Use partial fraction decomposition.

\[
\frac{1}{s(s^2 + 1)(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} + \frac{Ds + E}{s^2 + 4}
\]

Multiply both sides by \( s(s^2 + 1)(s^2 + 4) \).

\[
1 = A(s^2 + 1)(s^2 + 4) + (Bs + C)s(s^2 + 4) + (Ds + E)s(s^2 + 1)
\]

Plug in five random values for \( s \) to obtain a system of five equations for \( A, B, C, D, \) and \( E \).

\[
\begin{align*}
    s = 0 & : \quad 1 = 4A \\
    s = 1 & : \quad 1 = 10A + 5B + 5C + 2D + 2E \\
    s = 2 & : \quad 1 = 40A + 32B + 16C + 20D + 10E \\
    s = 3 & : \quad 1 = 130A + 117B + 39C + 90D + 30E \\
    s = 4 & : \quad 1 = 340A + 320B + 80C + 272D + 68E
\end{align*}
\]

Solving this system yields \( A = 1/4, B = -1/3, C = 0, D = 1/12, \) and \( E = 0 \).

\[
Y(s) = \left( \frac{1/4}{s} + \frac{-1/3 s}{s^2 + 1} + \frac{1/12 s}{s^2 + 4} \right) - \left( \frac{1/4}{s} + \frac{-1/3 s}{s^2 + 1} + \frac{1/12 s}{s^2 + 4} \right) e^{-\pi s}
\]

Now take the inverse Laplace transform of \( Y(s) \) to get \( y(t) \).

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\}
= \mathcal{L}^{-1}\left\{ \left( \frac{1/4}{s} + \frac{-1/3 s}{s^2 + 1} + \frac{1/12 s}{s^2 + 4} \right) - \left( \frac{1/4}{s} + \frac{-1/3 s}{s^2 + 1} + \frac{1/12 s}{s^2 + 4} \right) e^{-\pi s} \right\}
= \mathcal{L}^{-1}\left\{ \left( \frac{1/4}{s} + \frac{-1/3 s}{s^2 + 1} + \frac{1/12 s}{s^2 + 4} \right) \right\} - \mathcal{L}^{-1}\left\{ \left( \frac{1/4}{s} + \frac{-1/3 s}{s^2 + 1} + \frac{1/12 s}{s^2 + 4} \right) e^{-\pi s} \right\}
= \frac{1}{4} - \frac{1}{3} \cos t + \frac{1}{12} \cos 2t - \left[ \frac{1}{4} - \frac{1}{3} \cos(t - \pi) + \frac{1}{12} \cos 2(t - \pi) \right] H(t - \pi)
= \frac{1}{4} - \frac{1}{3} \cos t + \frac{1}{12} \cos 2t - \left( \frac{1}{4} + \frac{1}{3} \cos t + \frac{1}{12} \cos 2t \right) H(t - \pi)
= \frac{1}{12} (3 - 4 \cos t + \cos 2t) - \frac{1}{12} (3 + 4 \cos t + \cos 2t) H(t - \pi)
= \frac{1}{12} \left( 8 \sin^4 \frac{t}{2} \right) - \frac{1}{12} \left( 8 \cos^4 \frac{t}{2} \right) H(t - \pi)
= \frac{2}{3} \left[ \sin^4 \frac{t}{2} - \frac{1}{2} H(t - \pi) \cos^4 \frac{t}{2} \right]
= \frac{2}{3} \left[ \sin^4 \frac{t}{2} - u_\pi(t) \cos^4 \frac{t}{2} \right]
\]
Below is the graph of $y(t)$ versus $t$ superimposed on the graph of $f(t)$ versus $t$. 

$f = 1 - u_{\pi}(t)$