Problem 16

A certain spring-mass system satisfies the initial value problem

\[ u'' + \frac{1}{4}u' + u = kg(t), \quad u(0) = 0, \quad u'(0) = 0, \]

where \( g(t) = u_{3/2}(t) - u_{5/2}(t) \) and \( k > 0 \) is a parameter.

(a) Sketch the graph of \( g(t) \). Observe that it is a pulse of unit magnitude extending over one time unit.

(b) Solve the initial value problem.

(c) Plot the solution for \( k = 1/2, \ k = 1, \) and \( k = 2 \). Describe the principal features of the solution and how they depend on \( k \).

(d) Find, to two decimal places, the smallest value of \( k \) for which the solution \( u(t) \) reaches the value 2.

(e) Suppose \( k = 2 \). Find the time \( \tau \) after which \( |u(t)| < 0.1 \) for all \( t > \tau \).

Solution

Here is a plot of \( g(t) \) versus \( t \).

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( u(t) \) is defined here as

\[ U(s) = \mathcal{L}\{u(t)\} = \int_0^\infty e^{-st}u(t)\,dt. \]
Consequently, the first and second derivatives transform as follows.

\[ \mathcal{L}\left\{ \frac{du}{dt} \right\} = sU(s) - u(0) \]
\[ \mathcal{L}\left\{ \frac{d^2u}{dt^2} \right\} = s^2U(s) - su(0) - u'(0) \]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\left\{ u'' + \frac{1}{4}u' + u \right\} = \mathcal{L}\{kg(t)\} = \mathcal{L}\{k[u_{3/2}(t) - u_{5/2}(t)]\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{u''\} + \frac{1}{4}\mathcal{L}\{u'\} + \mathcal{L}\{u\} = k\mathcal{L}\{u_{3/2}(t)\} - k\mathcal{L}\{u_{5/2}(t)\} \]

\[ [s^2U(s) - su(0) - u'(0)] + \frac{1}{4}[sU(s) - u(0)] + [U(s)] = k\int_0^\infty e^{-st}[u_{3/2}(t)] \, dt - k\int_0^\infty e^{-st}[u_{5/2}(t)] \, dt \]

Plug in the initial conditions, \( u(0) = 0 \) and \( u'(0) = 0 \).

\[ [s^2U(s)] + \frac{1}{4}[sU(s)] + [U(s)] = k\int_{3/2}^\infty e^{-st} \, dt - k\int_{5/2}^\infty e^{-st} \, dt \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( U(s) \), the transformed solution.

\[ \left( s^2 + \frac{1}{4}s + 1 \right) U(s) = k \left( \frac{1}{s}e^{-st} \right) \bigg|_{3/2}^\infty - k \left( \frac{1}{s}e^{-st} \right) \bigg|_{5/2}^\infty \]

\[ = k \left( \frac{1}{s}e^{-3s/2} \right) - k \left( \frac{1}{s}e^{-5s/2} \right) \]

Solve for \( U(s) \) and write the right side in terms of known transforms.

\[ U(s) = \frac{k}{s(s^2 + \frac{1}{4} s + 1)} e^{-3s/2} - \frac{k}{s(s^2 + \frac{1}{4} s + 1)} e^{-5s/2} \]

Use partial fraction decomposition.

\[ \frac{k}{s(s^2 + \frac{1}{4} s + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \frac{1}{4} s + 1} \]

Multiply both sides by \( s(s^2 + \frac{1}{4} s + 1) \).

\[ k = A \left( s^2 + \frac{1}{4}s + 1 \right) + (Bs + C)s \]

Plug in three random values for \( s \) to get a system of three equations for \( A, B, \) and \( C \).

\[ s = 0 : \quad k = A \]
\[ s = 1 : \quad k = \frac{9}{4} A + B + C \]
\[ s = 2 : \quad k = \frac{11}{2} A + 4B + 2C \]
Solving this system yields $A = k$, $B = -k$, and $C = -k/4$.

$$U(s) = \left(\frac{k}{s} + \frac{-ks - k}{4} \right) e^{-3s/2} - \left(\frac{k}{s} + \frac{-ks - k}{4} \right) e^{-5s/2}$$

Complete the square in the denominators.

$$U(s) = \left(\frac{k}{s} + \frac{-ks - k}{4} \right) e^{-3s/2} - \left(\frac{k}{s} + \frac{-ks - k}{4} \right) e^{-5s/2}$$

$$U(s) = \left[ \frac{k}{s} - k \frac{s + \frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} \right] e^{-3s/2} - \left[ \frac{k}{s} - k \frac{s + \frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} \right] e^{-5s/2}$$

Make it so that $s + \frac{1}{8}$ appears in the numerators.

$$U(s) = \left[ \frac{k}{s} - k \frac{s + \frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} \right] e^{-3s/2} - \left[ \frac{k}{s} - k \frac{s + \frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} \right] e^{-5s/2}$$

Now take the inverse Laplace transform of $U(s)$ to get $u(t)$.

$$u(t) = \mathcal{L}^{-1}\{U(s)\}$$

$$u(t) = \mathcal{L}^{-1}\left\{ \left[ \frac{k}{s} - k \frac{s + \frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} \right] e^{-3s/2} - \left[ \frac{k}{s} - k \frac{s + \frac{1}{8}}{(s + \frac{1}{8})^2 + \frac{63}{64}} \right] e^{-5s/2} \right\}$$

Below are several plots of $u(t)$ versus $t$ for several values of $k$. 

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The first maximum of $u$ seems to be proportional to $k$; the maximum is 2 when $k \approx 2.51$. The solution starts to oscillate at $t = 3/2$ and has an amplitude that decays with time.
Adjust the lower and upper bounds of the $k = 2$ graph to $-0.1$ and $0.1$, respectively, to find the time at which the amplitude is smaller than $0.1$ for all later times.