Problem 17

Modify the problem in Example 2 of this section by replacing the given forcing function $g(t)$ by

$$f(t) = \left[u_5(t)(t - 5) - u_{5+k}(t)(t - 5 - k)\right]/k.$$ 

(a) Sketch the graph of $f(t)$ and describe how it depends on $k$. For what value of $k$ is $f(t)$ identical to $g(t)$ in the example?

(b) Solve the initial value problem

$$y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$ 

(c) The solution in part (b) depends on $k$, but for sufficiently large $t$ the solution is always a simple harmonic oscillation about $y = 1/4$. Try to decide how the amplitude of this eventual oscillation depends on $k$. Then confirm your conclusion by plotting the solution for a few different values of $k$.

Solution

Part (a)

The value of $k$ determines the horizontal length of the ramp. $f(t)$ is identical to $g(t)$ if $k = 5$. 

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Part (b)

\[ y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 0. \]

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t) \, dt. \]

Consequently, the first and second derivatives transform as follows.

\[ \mathcal{L}\left\{ \frac{dy}{dt} \right\} = sY(s) - y(0) \]

\[ \mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} = s^2Y(s) - sy(0) - y'(0) \]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\{y'' + 4y\} = \mathcal{L}\{f(t)\} = \mathcal{L}\left\{ \frac{1}{k}[u_5(t)(t - 5) - u_{5+k}(t)(t - 5 - k)] \right\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \frac{1}{k}\mathcal{L}\{u_5(t)(t - 5)\} - \frac{1}{k}\mathcal{L}\{u_{5+k}(t)(t - 5 - k)\} \]

\[ [s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] = \frac{1}{k}\int_0^\infty e^{-st}[u_5(t)(t - 5)] \, dt - \frac{1}{k}\int_0^\infty e^{-st}[u_{5+k}(t)(t - 5 - k)] \, dt \]

Plug in the initial conditions, \( y(0) = 0 \) and \( y'(0) = 0 \).

\[ [s^2Y(s)] + 4[Y(s)] = \frac{1}{k}\int_5^\infty (t - 5)e^{-st} \, dt - \frac{1}{k}\int_{5+k}^\infty (t - 5 - k)e^{-st} \, dt \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[ (s^2 + 4)Y(s) = \frac{1}{k}\left( \frac{e^{-5s}}{s^2} \right) - \frac{1}{k}\left[ \frac{e^{-(5+k)s}}{s^2} \right] \]

Solve for \( Y(s) \) and write the right side in terms of known transforms.

\[ Y(s) = \frac{1}{k}\left[ \frac{1}{s^2(s^2 + 4)}e^{-5s} - \frac{1}{s^2(s^2 + 4)}e^{-(5+k)s} \right] \]

\[ = \frac{1}{k}\left[ \left( \frac{1}{s^2} + \frac{-1}{s^2 + 4} \right)e^{-5s} - \left( \frac{1}{s^2} + \frac{-1}{s^2 + 4} \right)e^{-(5+k)s} \right] \]

\[ = \frac{1}{k}\left( \frac{1}{s^2} - \frac{1}{s^2 + 4} \right)e^{-5s} - \frac{1}{k}\left( \frac{1}{s^2} - \frac{1}{s^2 + 4} \right)e^{-(5+k)s} \]

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Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

\[ y(t) = \mathcal{L}^{-1}\{Y(s)\} \]

\[ = \mathcal{L}^{-1}\left\{ \frac{1}{k} \left( \frac{1}{s^2} - \frac{1}{8} \frac{2}{s^2 + 4} \right) e^{-5s} - \frac{1}{k} \left( \frac{1}{s^2} - \frac{1}{8} \frac{2}{s^2 + 4} \right) e^{-(5+k)s} \right\} \]

\[ = \frac{1}{k} \mathcal{L}^{-1}\left\{ \left( \frac{1}{s^2} - \frac{1}{8} \frac{2}{s^2 + 4} \right) e^{-5s} \right\} - \frac{1}{k} \mathcal{L}^{-1}\left\{ \left( \frac{1}{s^2} - \frac{1}{8} \frac{2}{s^2 + 4} \right) e^{-(5+k)s} \right\} \]

\[ = \frac{1}{k} \left\{ \frac{1}{4}(t - 5) - \frac{1}{8} \sin[2(t - 5)] \right\} H(t - 5) - \frac{1}{k} \left\{ \frac{1}{4}(t - 5 - k) - \frac{1}{8} \sin[2(t - 5 - k)] \right\} H(t - 5 - k) \]

\[ = \frac{1}{k} \left\{ \frac{1}{4}(t - 5) - \frac{1}{8} \sin[2(t - 5)] \right\} u_5(t) - \frac{1}{k} \left\{ \frac{1}{4}(t - 5 - k) - \frac{1}{8} \sin[2(t - 5 - k)] \right\} u_{5+k}(t) \]

**Part (c)**

Below are several plots of $y(t)$ versus $t$ for several values of $k$. 

![Graph of y(t) versus t with different values of k](image-url)
For $k = 1$, $A \approx 0.208$.

For $k = \frac{3}{2}$, $A \approx 0.165$. 
For $k = 3$:

$$A = 0.0114$$

For $k = 3.5$:

$$A \approx 0.0246$$
The graphs above show the behavior of solutions for different values of $k$.

For $k = 4$, the amplitude $A$ is approximately 0.0462.

For $k = 4.5$, the amplitude $A$ is approximately 0.0532.
$k = 6$

$A \approx 0.0101$

$\frac{dy}{dt} = ky - Ay^2$

$y(0) = 0$

$k = 6.5$

$A \approx 0.0073$
$k = 10$
$A \approx 0.0126$

$y$

$t$

$y$

$t$

$k = 10.5$
$A \approx 0.0199$
For the given boundary value problem, we plot two graphs for different values of the parameter $k$.

**Graph 1:**
- For $k = 12$, the initial value of $y$ is approximately 0.0102.

**Graph 2:**
- For $k = 12.5$, the initial value of $y$ is approximately 0.0009.

These graphs illustrate the behavior of the solution as the parameter $k$ changes.
\( k = 13 \)
\( A \approx 0.0071 \)

\( k = 13.5 \)
\( A \approx 0.0134 \)
Now plot the values of $A$ versus the values of $k$ to determine the functional dependence.