Problem 19

Consider the initial value problem

\[ y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \]

where

\[ f(t) = u_0(t) + 2 \sum_{k=1}^{n} (-1)^k u_{k\pi}(t). \]

(a) Draw the graph of \( f(t) \) on an interval such as \( 0 \leq t \leq 6\pi \).

(b) Find the solution of the initial value problem.

(c) Let \( n = 15 \) and plot the graph of the solution for \( 0 \leq t \leq 60 \). Describe the solution and explain why it behaves as it does.

(d) Investigate how the solution changes as \( n \) increases. What happens as \( n \to \infty \)?

Solution

On the interval \( 0 \leq t \leq 6\pi \), \( u_{k\pi}(t) \) is nonzero if \( k < 6 \) and 0 if \( k \geq 6 \).

\[
\begin{align*}
f(t) &= u_0(t) + 2 \sum_{k=1}^{5} (-1)^k u_{k\pi}(t) + 2 \sum_{k=6}^{n} (-1)^k u_{k\pi}(t) \\
&= u_0(t) + 2 \sum_{k=1}^{5} (-1)^k u_{k\pi}(t) + 2 \sum_{k=6}^{n} (-1)^k(0) \\
&= u_0(t) + 2 \sum_{k=1}^{5} (-1)^k u_{k\pi}(t)
\end{align*}
\]
Because the ODE is linear, the Laplace transform can be used to solve it. The Laplace transform of a function \( y(t) \) is defined to be

\[
Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt.
\]

Consequently, the first and second derivatives transform as follows.

\[
\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)
\]

\[
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)
\]

Substitute the provided function for \( f(t) \) and take the Laplace transform of both sides of the ODE.

\[
\mathcal{L}\{y'' + y\} = \mathcal{L}\left\{u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)\right\}
\]

Use the fact that the transform is a linear operator.

\[
\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_0(t)\} + 2\sum_{k=1}^{n}(-1)^k \mathcal{L}\{u_{k\pi}(t)\}
\]

\[
[s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^\infty e^{-st}u_0(t)\,dt + 2\sum_{k=1}^{n}(-1)^k \int_0^\infty e^{-st}u_{k\pi}(t)\,dt
\]

Plug in the initial conditions, \( y(0) = 0 \) and \( y'(0) = 0 \).

\[
[s^2Y(s)] + Y(s) = \int_0^\infty e^{-st} dt + 2\sum_{k=1}^{n}(-1)^k \int_{k\pi}^\infty e^{-st} dt
\]

\[
(s^2 + 1)Y(s) = \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k \left( \frac{e^{-k\pi s}}{s} \right)
\]

Solve for \( Y(s) \).

\[
Y(s) = \frac{1}{s(s^2 + 1)} + 2\sum_{k=1}^{n}(-1)^k \left[ \frac{1}{s(s^2 + 1)} \right] e^{-k\pi s}
\]

Now write it in terms of known transforms by using partial fraction decomposition.

\[
Y(s) = \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) + 2\sum_{k=1}^{n}(-1)^k \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-k\pi s}
\]
Take the inverse Laplace transform of \( Y(s) \) to get \( y(t) \).

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\} \\
= \mathcal{L}^{-1}\left\{ \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) + 2 \sum_{k=1}^{n} (-1)^k \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-k\pi s} \right\} \\
= \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{s}{s^2 + 1} \right\} + 2 \sum_{k=1}^{n} (-1)^k \mathcal{L}^{-1}\left\{ \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-k\pi s} \right\} \\
= 1 - \cos t + 2 \sum_{k=1}^{n} (-1)^k \left[ 1 - \cos(t - k\pi) \right] H(t - k\pi) \\
= 1 - \cos t + 2 \sum_{k=1}^{n} (-1)^k \left[ 1 - \cos(t - k\pi) \right] u_{k\pi}(t)
\]

Graphs of \( y(t) \) versus \( t \) are shown below for \( n = 1, n = 2, n = 5, n = 10, \) and \( n = 15 \).
The value of $n$ determines the number of ridges as the graph increases. The graph oscillates with an amplitude of 1 and a period of $2\pi$ about the value $y = 2n + 1$. If $n \to \infty$, the graph grows at a linear rate forever.