Problem 3

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

\[
y'' + 4y = \sin t - u_{2\pi}(t) \sin(t - 2\pi); \quad y(0) = 0, \quad y'(0) = 0
\]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \(y(t)\) is defined here as

\[
Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt.
\]

Consequently, the first and second derivatives transform as follows.

\[
\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)
\]
\[
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)
\]

Apply the Laplace transform to both sides of the ODE.

\[
\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\sin t - u_{2\pi}(t) \sin(t - 2\pi)\}
\]

Use the fact that the transform is a linear operator.

\[
\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{u_{2\pi}(t) \sin(t - 2\pi)\}
\]

\[
[s^2Y(s) - sy(0) - y'(0)] + 4[Y(s)] = \frac{1}{s^2 + 1} - \frac{1}{s^2 + 1} e^{-2\pi s}
\]

Plug in the initial conditions, \(y(0) = 0\) and \(y'(0) = 0\).

\[
[s^2Y(s)] + 4[Y(s)] = \frac{1}{s^2 + 1} - \frac{1}{s^2 + 1} e^{-2\pi s}
\]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \(Y(s)\), the transformed solution.

\[
(s^2 + 4)Y(s) = \frac{1}{s^2 + 1} - \frac{1}{s^2 + 1} e^{-2\pi s}
\]

\[
Y(s) = \frac{1}{(s^2 + 1)(s^2 + 4)} - \frac{1}{(s^2 + 1)(s^2 + 4)} e^{-2\pi s}
\]

In order to write \(Y(s)\) in terms of known transforms, use partial fraction decomposition.

\[
\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}
\]

www.stemjock.com
Multiply both sides by \((s^2 + 1)(s^2 + 4)\).

\[ 1 = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1) \]

Plug in four random values of \(s\) to get a system of equations for \(A\), \(B\), \(C\), and \(D\).

\[
\begin{align*}
  s = 0 : & \quad 1 = 4B + D \\
  s = 1 : & \quad 1 = 5A + 5B + 2C + 2D \\
  s = 2 : & \quad 1 = 16A + 8B + 10C + 5D \\
  s = 3 : & \quad 1 = 39A + 13B + 30C + 10D
\end{align*}
\]

Solving this system yields \(A = 0\), \(B = 1/3\), \(C = 0\), and \(D = -1/3\).

\[
Y(s) = \frac{\frac{1}{3}}{s^2 + 1} + \frac{\frac{-1}{3}}{s^2 + 4} - \left( \frac{\frac{1}{3}}{s^2 + 1} + \frac{\frac{-1}{3}}{s^2 + 4} \right) e^{-2\pi s}
\]

\[
= \frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} - \left( \frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} \right) e^{-2\pi s}
\]

Now take the inverse Laplace transform of \(Y(s)\) to get \(y(t)\).

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1} \left\{ \frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} - \left( \frac{1}{3} \frac{1}{s^2 + 1} - \frac{1}{6} \frac{2}{s^2 + 4} \right) e^{-2\pi s} \right\}
\]

\[
= \frac{1}{3} \sin t - \frac{1}{6} \sin 2t - \left[ \frac{1}{3} \sin(t - 2\pi) - \frac{1}{6} \sin 2(t - 2\pi) \right] H(t - 2\pi)
\]

\[
= \frac{1}{6} (2\sin t - \sin 2t) - \frac{1}{6} [2\sin(t - 2\pi) - \sin 2(t - 2\pi)] H(t - 2\pi)
\]

\[
= \frac{1}{6} (2\sin t - \sin 2t) - \frac{1}{6} (2\sin t - \sin 2t) u_{2\pi}(t)
\]

\[
= \frac{1}{6} (2\sin t - 2\sin t \cos t) - \frac{1}{6} (2\sin t - 2\sin t \cos t) u_{2\pi}(t)
\]

\[
= \frac{1}{3} \sin t(1 - \cos t) - \frac{1}{3} \sin t(1 - \cos t) u_{2\pi}(t)
\]

\[
= \frac{1}{3} \sin t(1 - \cos t)[1 - u_{2\pi}(t)]
\]
Below is the graph of $y(t)$ versus $t$ superimposed on the graph of $f(t)$ versus $t$.

$$f = \sin t - u_2 \pi(t) \sin(t - 2 \pi)$$