Problem 7

In each of Problems 1 through 13:

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

\[ y'' + y = u_{3\pi}(t); \quad y(0) = 1, \quad y'(0) = 0 \]

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function \( y(t) \) is defined here as

\[ Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st}y(t)\,dt. \]

Consequently, the first and second derivatives transform as follows.

\[ \mathcal{L}\left\{ \frac{dy}{dt} \right\} = sY(s) - y(0) \]
\[ \mathcal{L}\left\{ \frac{d^2y}{dt^2} \right\} = s^2Y(s) - sy(0) - y'(0) \]

Apply the Laplace transform to both sides of the ODE.

\[ \mathcal{L}\{y'' + y\} = \mathcal{L}\{u_{3\pi}(t)\} \]

Use the fact that the transform is a linear operator.

\[ \mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_{3\pi}(t)\} \]

\[ [s^2Y(s) - sy(0) - y'(0)] + Y(s) = \int_0^\infty e^{-st}[u_{3\pi}(t)]\,dt \]

Plug in the initial conditions, \( y(0) = 1 \) and \( y'(0) = 0 \).

\[ [s^2Y(s) - s] + Y(s) = \int_{3\pi}^\infty e^{-st}\,dt \]

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for \( Y \), the transformed solution.

\[ (s^2 + 1)Y(s) - s = \left( \frac{1}{s} e^{-3\pi s} \right) \bigg|_{3\pi}^{\infty} \]
   \[ = \frac{1}{s} e^{-3\pi s} \]
   \[ (s^2 + 1)Y(s) = \frac{1}{s} e^{-3\pi s} + s \]

\[ Y(s) = \frac{1}{s(s^2 + 1)} e^{-3\pi s} + s \]
   \[ = \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-3\pi s} + \frac{s}{s^2 + 1} \]
Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{ \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-3\pi s} + \frac{s}{s^2 + 1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-3\pi s} \right\} + \mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 1} \right\}$$

$$= [1 - \cos(t - 3\pi)]H(t - 3\pi) + \cos t$$

$$= [1 - \cos(t - \pi)]H(t - 3\pi) + \cos t$$

$$= (1 + \cos t)H(t - 3\pi) + \cos t$$

$$= (1 + \cos t)u_{3\pi}(t) + \cos t$$

Below is the graph of $y(t)$ versus $t$ superimposed on the graph of $f(t)$ versus $t$.

![Graph of y(t) and f(t) superimposed](www.stemjock.com)