Problem 3

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t); \qquad y(0) = 0, \quad y'(0) = 1/2$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{\delta(t-5) + u_{10}(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-5)\} + \mathcal{L}\{u_{10}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2[Y(s)] = \int_0^\infty e^{-st}[\delta(t-5)]\, dt + \int_0^\infty e^{-st}[u_{10}(t)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1/2$.

$$[s^2Y(s) - 1/2] + 3[sY(s)] + 2[Y(s)] = e^{-s(5)} + \int_{10}^\infty e^{-st}\, dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 3s + 2)Y(s) - 1/2 = e^{-5s} + \frac{e^{-10s}}{s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned} Y(s) &= \frac{1/2}{s^2+3s+2} + \frac{1}{s^2+3s+2}e^{-5s} + \frac{1}{s(s^2+3s+2)}e^{-10s} \\ &= \frac{1/2}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)}e^{-5s} + \frac{1}{s(s+2)(s+1)}e^{-10s} \end{aligned}$$

Use partial fraction decomposition.

$$Y(s) = \left(\frac{1/2}{s+1} - \frac{1/2}{s+2}\right) + \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-5s} + \left(\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\right)e^{-10s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{1/2}{s+1} - \frac{1/2}{s+2}\right) + \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-5s} + \left(\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\right)e^{-10s}\right\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{1/2}{s+1} - \frac{1/2}{s+2}\right)\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-5s}\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\right)e^{-10s}\right\} \\
&= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + \left[e^{-(t-5)} - e^{-2(t-5)}\right]H(t-5) + \left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right]H(t-10) \\
&= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + \left[e^{-(t-5)} - e^{-2(t-5)}\right]u_5(t) + \left[\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)}\right]u_{10}(t)
\end{aligned}$$

Below is a plot of $y(t)$ versus t.