Problem 5

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' + 2y' + 3y = \sin t + \delta(t - 3\pi); \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 2y' + 3y\} = \mathcal{L}\{\sin t + \delta(t - 3\pi)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{\delta(t - 3\pi)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 3[Y(s)] = \frac{1}{s^2+1} + \int_0^\infty e^{-st}[\delta(t - 3\pi)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + 2[sY(s)] + 3[Y(s)] = \frac{1}{s^2+1} + e^{-s(3\pi)}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 2s + 3)Y(s) = \frac{1}{s^2+1} + e^{-3\pi s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$Y(s) = \frac{1}{(s^2+1)(s^2+2s+3)} + \frac{1}{s^2+2s+3}e^{-3\pi s}$$

Use partial fraction decomposition.

$$\frac{1}{(s^2+1)(s^2+2s+3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+3}$$

Multiply both sides by $(s^2+1)(s^2+2s+3)$.

$$1 = (As+B)(s^2+2s+3) + (Cs+D)(s^2+1)$$

Plug in four random values for s to get a system of four equations for A, B, C, and D.

$$\begin{aligned}
s=0:&\quad 1 = 3B + D\\
s=1:&\quad 1 = 6A + 6B + 2C + 2D\\
s=2:&\quad 1 = 22A + 11B + 10C + 5D\\
s=3:&\quad 1 = 54A + 18B + 30C + 10D
\end{aligned}$$

Solving this system yields $A=-1/4$, $B=1/4$, $C=1/4$, and $D=1/4$.

$$Y(s) = \frac{-\frac{1}{4}s+\frac{1}{4}}{s^2+1} + \frac{\frac{1}{4}s+\frac{1}{4}}{s^2+2s+3} + \frac{1}{s^2+2s+3}e^{-3\pi s}$$

Complete the square in the denominators.

$$\begin{aligned}
Y(s) &= \frac{-\frac{1}{4}s+\frac{1}{4}}{s^2+1} + \frac{\frac{1}{4}s+\frac{1}{4}}{s^2+2s+1+3-1} + \frac{1}{s^2+2s+1+3-1}e^{-3\pi s}\\
&= \frac{-\frac{1}{4}s+\frac{1}{4}}{s^2+1} + \frac{\frac{1}{4}s+\frac{1}{4}}{(s+1)^2+2} + \frac{1}{(s+1)^2+2}e^{-3\pi s}\\
&= -\frac{1}{4}\frac{s}{s^2+1} + \frac{1}{4}\frac{1}{s^2+1} + \frac{1}{4}\frac{s+1}{(s+1)^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2}e^{-3\pi s}
\end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\}\\
&= \mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{s}{s^2+1} + \frac{1}{4}\frac{1}{s^2+1} + \frac{1}{4}\frac{s+1}{(s+1)^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2}e^{-3\pi s}\right\}\\
&= -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2}\right\} + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+2}e^{-3\pi s}\right\}\\
&= -\frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{4}e^{-t}\cos\sqrt{2}t + \frac{1}{\sqrt{2}}e^{-(t-3\pi)}\sin[\sqrt{2}(t-3\pi)]H(t-3\pi)\\
&= \frac{1}{4}(\sin t - \cos t + e^{-t}\cos\sqrt{2}t) + \frac{1}{\sqrt{2}}e^{3\pi-t}\sin[\sqrt{2}(t-3\pi)]u_{3\pi}(t)
\end{aligned}$$

Below is a plot of $y(t)$ versus t.