Problem 9

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y'' + y = u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t); \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{u_{\pi/2}(t) + 3\delta(t - 3\pi/2) - u_{2\pi}(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_{\pi/2}(t)\} + 3\mathcal{L}\{\delta(t - 3\pi/2)\} - \mathcal{L}\{u_{2\pi}(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + [Y(s)] = \int_0^\infty e^{-st}[u_{\pi/2}(t)]\,dt + 3\int_0^\infty e^{-st}[\delta(t - 3\pi/2)]\,dt - \int_0^\infty e^{-st}[u_{2\pi}(t)]\,dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$[s^2Y(s)] + [Y(s)] = \int_{\pi/2}^\infty e^{-st}\,dt + 3e^{-s(3\pi/2)} - \int_{2\pi}^\infty e^{-st}\,dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^2 + 1)Y(s) = \frac{e^{-\pi s/2}}{s} + 3e^{-3\pi s/2} - \frac{e^{-2\pi s}}{s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$Y(s) = \frac{1}{s(s^2 + 1)}e^{-\pi s/2} + 3\frac{1}{s^2 + 1}e^{-3\pi s/2} - \frac{1}{s(s^2 + 1)}e^{-2\pi s}$$

Use partial fraction decomposition.

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}$$

Multiply both sides by $s(s^2+1)$.

$$1 = A(s^2+1) + (Bs+C)s$$

Plug in three random values of s to get a system of three equations for A, B, and C.

$$\begin{aligned} s=0: &\quad 1 = A \\ s=1: &\quad 1 = 2A + B + C \\ s=2: &\quad 1 = 5A + 4B + 2C \end{aligned}$$

Solving this system yields $A = 1$, $B = -1$, and $C = 0$.

$$Y(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-\pi s/2} + 3\frac{1}{s^2+1}e^{-3\pi s/2} - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-2\pi s}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-\pi s/2} + 3\frac{1}{s^2+1}e^{-3\pi s/2} - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-2\pi s}\right\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-\pi s/2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-3\pi s/2}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-2\pi s}\right\} \\ &= [1-\cos(t-\pi/2)]H(t-\pi/2) + 3\sin(t-3\pi/2)H(t-3\pi/2) - [1-\cos(t-2\pi)]H(t-2\pi) \\ &= (1-\sin t)H(t-\pi/2) + 3(\cos t)H(t-3\pi/2) - (1-\cos t)H(t-2\pi) \\ &= u_{\pi/2}(t)(1-\sin t) + 3u_{3\pi/2}(t)\cos t - u_{2\pi}(t)(1-\cos t) \end{aligned}$$

Below is a plot of $y(t)$ versus t.