Problem 12

In each of Problems 1 through 12:

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

$$y^{(4)} - y = \delta(t-1); \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the derivatives transform as follows.

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y^{(4)} - y\} = \mathcal{L}\{\delta(t-1)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-1)\}$$

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] - [Y(s)] = \int_0^\infty e^{-st}[\delta(t-1)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$ and $y''(0) = 0$ and $y'''(0) = 0$.

$$[s^4Y(s)] - [Y(s)] = e^{-s(1)}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(s^4 - 1)Y(s) = e^{-s}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned}
Y(s) &= \frac{1}{s^4 - 1} e^{-s} \\
&= \frac{1}{(s-1)(s+1)(s^2+1)} e^{-s} \\
&= \left(\frac{\frac{1}{4}}{s-1} + \frac{-\frac{1}{4}}{s+1} + \frac{-\frac{1}{2}}{s^2+1} \right) e^{-s}
\end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s-1} + \frac{-\frac{1}{4}}{s+1} + \frac{-\frac{1}{2}}{s^2+1}\right)e^{-s}\right\} \\
&= \left[\frac{1}{4}e^{(t-1)} - \frac{1}{4}e^{-(t-1)} - \frac{1}{2}\sin(t-1)\right] H(t-1) \\
&= \frac{1}{2}\left[\frac{e^{(t-1)} - e^{-(t-1)}}{2} - \sin(t-1)\right] H(t-1) \\
&= \frac{1}{2}[\sinh(t-1) - \sin(t-1)]H(t-1) \\
&= \frac{1}{2}[\sinh(t-1) - \sin(t-1)]u_1(t)
\end{aligned}$$

Below is a plot of $y(t)$ versus t.