Problem 13

Consider again the system in Example 1 of this section, in which an oscillation is excited by a unit impulse at $t = 5$. Suppose that it is desired to bring the system to rest again after exactly one cycle—that is, when the response first returns to equilibrium moving in the positive direction.

(a) Determine the impulse $k\delta(t - t_0)$ that should be applied to the system in order to accomplish this objective. Note that k is the magnitude of the impulse and t_0 is the time of its application.

(b) Solve the resulting initial value problem, and plot its solution to confirm that it behaves in the specified manner.

Solution

The system in Example 1 is governed by the initial value problem,

$$2y'' + y' + 2y = \delta(t - 5), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solving this problem with the Laplace transform yields

$$y(t) = \frac{2}{\sqrt{15}} e^{-(t-5)/4} \sin\left[\frac{\sqrt{15}}{4}(t - 5)\right] H(t - 5).$$

Below is a plot of $y(t)$ versus t.

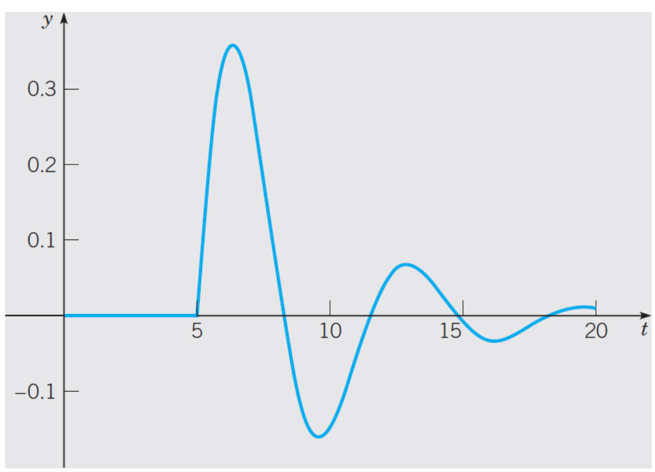

FIGURE 6.5.3 Solution of the initial value problem (17), (18): $2y'' + y' + 2y = \delta(t - 5)$, $y(0) = 0$, $y'(0) = 0$.

The new impulse needs to be applied after one oscillation has been completed. Use this fact to determine t_0.

$$\frac{\sqrt{15}}{4}(t - 5) = 2\pi \quad \to \quad t_0 = 5 + \frac{8\pi}{\sqrt{15}}$$

k will be found later. For the moment, solve the initial value problem with the second applied impulse.

$$2y'' + y' + 2y = \delta(t - 5) + k\delta(t - t_0), \qquad y(0) = 0, \quad y'(0) = 0.$$

The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{2y'' + y' + 2y\} = \mathcal{L}\{\delta(t-5) + k\delta(t-t_0)\}$$

Use the fact that the transform is a linear operator.

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-5)\} + k\mathcal{L}\{\delta(t-t_0)\}$$

$$2[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] + 2[Y(s)] = \int_0^\infty e^{-st}[\delta(t-5)]\, dt + k\int_0^\infty e^{-st}[\delta(t-t_0)]\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$.

$$2[s^2Y(s)] + [sY(s)] + 2[Y(s)] = e^{-s(5)} + ke^{-s(t_0)}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$(2s^2 + s + 2)Y(s) = e^{-5s} + ke^{-st_0}$$

Solve for $Y(s)$ and write it in terms of known transforms.

$$\begin{aligned}
Y(s) &= \frac{1}{2s^2+s+2}e^{-5s} + k\frac{1}{2s^2+s+2}e^{-st_0} \\
&= \frac{1}{2}\frac{1}{s^2+\frac{1}{2}s+1}e^{-5s} + \frac{k}{2}\frac{1}{s^2+\frac{1}{2}s+1}e^{-st_0} \\
&= \frac{1}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}e^{-5s} + \frac{k}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}e^{-st_0} \\
&= \frac{2}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}e^{-5s} + \frac{2k}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}e^{-st_0}
\end{aligned}$$

Now take the inverse Laplace transform of $Y(s)$ to get $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{2}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}e^{-5s} + \frac{2k}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}e^{-st_0}\right\} \\
&= \frac{2}{\sqrt{15}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}e^{-5s}\right\} + \frac{2k}{\sqrt{15}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}e^{-st_0}\right\} \\
&= \frac{2}{\sqrt{15}}e^{-(t-5)/4}\sin\left[\frac{\sqrt{15}}{4}(t-5)\right]H(t-5) + \frac{2k}{\sqrt{15}}e^{-(t-t_0)/4}\sin\left[\frac{\sqrt{15}}{4}(t-t_0)\right]H(t-t_0)
\end{aligned}$$

Once t takes the value of t_0, the amplitude of the first term is

$$\frac{2}{\sqrt{15}}e^{-(t_0-5)/4} = \frac{2}{\sqrt{15}}e^{-2\pi/\sqrt{15}},$$

and the amplitude of the second term is

$$\frac{2k}{\sqrt{15}}e^{-(t_0-t_0)/4} = \frac{2k}{\sqrt{15}}.$$

For the second term to cancel the first one, we require that

$$\frac{2k}{\sqrt{15}} = -\frac{2}{\sqrt{15}}e^{-2\pi/\sqrt{15}} \quad \rightarrow \quad k = -e^{-2\pi/\sqrt{15}}.$$

With this value of k and the value found for t_0, the graph of $y(t)$ versus t displays the desired result.